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7

Structural Theory

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7.1 Introduction

In this chapter, general forms of three sets of equations required in solving a solid mechanics problem and their extensions into structural theory are presented. In particular, a more generally used method, displacement method, is expressed in detail.

7.1.1 Basic Equations: Equilibrium, Compatibility, and Constitutive Law

In general, solving a solid mechanics problem must satisfy equations of equilibrium (static or dynamic), conditions of compatibility between strains and displacements, and stress–strain relations or material constitutive law (see Figure 7.1). The initial and boundary conditions on forces and displacements are naturally included.

From consideration of equilibrium equations, one can relate the stresses inside a body to external excitations, including body and surface forces. There are three equations of equilibrium relating the

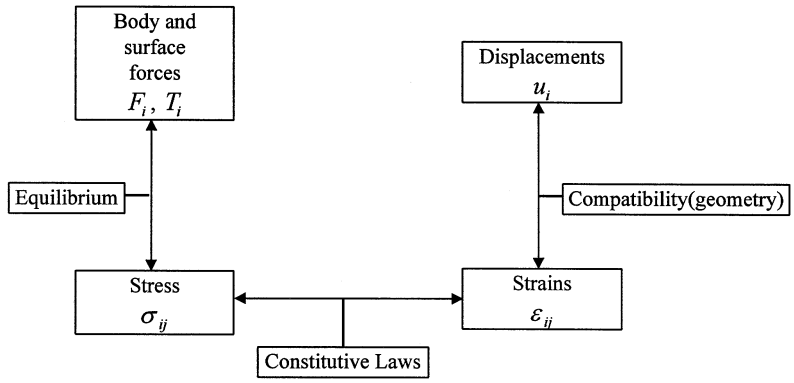


FIGURE 7.1 Relations of variables in solving a solid mechanics problem.

six components of stress tensor σ_{ij} for an infinitesimal material element which will be shown later in Section 7.2.1. In the case of dynamics, the equilibrium equations are replaced by equations of motion, which contain second-order derivatives of displacement with respect to time.

In the same way, taking into account geometric conditions, one can relate strains inside a body to its displacements, by six equations of kinematics expressing the six components of strain (ε_{ij}) in terms of the three components of displacement (u_i). These are known as the strain–displacement relations (see Section 7.3.1).

Both the equations of equilibrium and kinematics are valid regardless of the specific material of which the body is made. The influence of the material is expressed by constitutive laws in six equations. In the simplest case, not considering the effects of temperature, time, loading rates, and loading paths, these can be described by relations between stress and strain only.

Six stress components, six strain components, and three displacement components are connected by three equilibrium equations, six kinematics equations, and six constitutive equations. The 15 unknown quantities can be determined from the system of 15 equations.

It should be pointed out that the principle of superposition is valid only when small deformations and elastic materials are assumed.

7.1.2 Three Levels: Continuous Mechanics, Finite–Element Method, Beam–Column Theory

In solving a solid mechanics problem, the most direct method solves the three sets of equations described in the previous section. Generally, there are three ways to establish the basic unknowns, namely, the displacement components, the stress components, or a combination of both. The corresponding procedures are called the displacement method, the stress method, or the mixed method, respectively. But these direct methods are only practicable in some simple circumstances, such as those detailed in elastic theory of solid mechanics.

Many complex problems cannot be easily solved with conventional procedures. Complexities arise due to factors such as irregular geometry, nonhomogeneities, nonlinearity, and arbitrary loading conditions. An alternative now available is based on a concept of discretization. The finite-element method (FEM) divides a body into many "small" bodies called finite elements. Formulations by the FEM on the laws and principles governing the behavior of the body usually result in a set of simultaneous equations that can be solved by direct or iterative procedures. And loading effects such as deformations and stresses can be evaluated within certain accuracy. Up to now, FEM has been the most widely used structural analysis method.

In dealing with a continuous beam, the size of the three sets of equations is greatly reduced by assuming characteristics of beam members such as plane sections remain plane. For framed structures

or structures constructed using beam–columns, structural mechanics gives them a more pithy and practical analysis.

7.1.3 Theoretical Structural Mechanics, Computational Structural Mechanics, and Qualitative Structural Mechanics

Structural mechanics deals with a system of members connected by joints which may be pinned or rigid. Classical methods of structural analysis are based on principles such as the principle of virtual displacement, the minimization of total potential energy, the minimization of total complementary energy, which result in the three sets of governing equations. Unfortunately, conventional methods are generally intended for hand calculations and developers of the FEM took great pains to minimize the amount of calculations required, even at the expense of making the methods somewhat unsystematic. This made the conventional methods unattractive for translation to computer codes.

The digital computer called for a more systematic method of structural analysis, leading to computational structural mechanics. By taking great care to formulate the tools of matrix notation in a mathematically consistent fashion, the analyst achieved a systematic approach convenient for automatic computation: matrix analysis of structures. One of the hallmarks of structural matrix analysis is its systematic nature, which renders digital computers even more important in structural engineering.

Of course, the analyst must maintain a critical, even skeptical, attitude toward computer results. In any event, computer results must satisfy our intuition of what is "reasonable." This qualitative judgment requires that the analyst possess a full understanding of structural behavior, both that being modeled by the program and that which can be expected in the actual structures. Engineers should decide what approximations are reasonable for the particular structure and verify that these approximations are indeed valid, and know how to design the structure so that its behavior is in reasonable agreement with the model adopted to analyze it. This is the main task of a structural analyst.

7.1.4 Matrix Analysis of Structures: Force Method and Displacement Method

Matrix analysis of structures was developed in the early 1950s. Although it was initially used on fuselage analysis, this method was proved to be pertinent to any complex structure. If internal forces are selected as basic unknowns, the analysis method is referred to as force method; in a similar way, the displacement method refers to the case where displacements are selected as primary unknowns. Both methods involve obtaining the joint equilibrium equations in terms of the basic internal forces or joint displacements as primary unknowns and solving the resulting set of equations for these unknowns. Having done this, one can obtain internal forces by backsubstitution, since even in the case of the displacement method the joint displacements determine the basic displacements of each member, which are directly related to internal forces and stresses in the member.

A major feature evident in structural matrix analysis is an emphasis on a systematic approach to the statement of the problem. This systematic characteristic together with matrix notation makes it especially convenient for computer coding. In fact, the displacement method, whose basic unknowns are uniquely defined, is generally more convenient than the force method. Most general-purpose structural analysis programs are displacement based. But there are still cases where it may be more desirable to use the force method.

7.2 Equilibrium Equations

7.2.1 Equilibrium Equation and Virtual Work Equation

For any volume V of a material body having A as surface area, as shown in Figure 7.2, it has the following conditions of equilibrium:

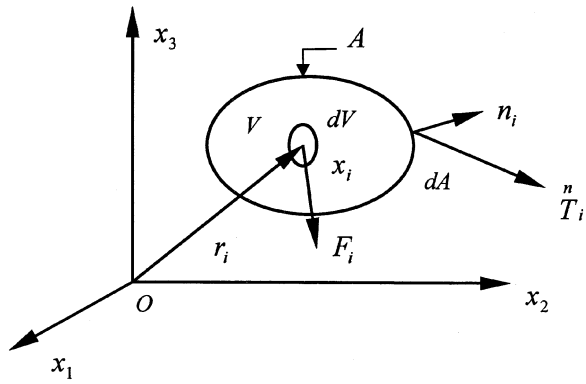


FIGURE 7.2 Derivation of equations of equilibrium.

At surface points

$$T_i = \sigma_{ji} n_j \tag{7.1a}$$

At internal points

$$\sigma_{ji,j} + F_i = 0 \tag{7.1b}$$

$$\sigma_{ji} = \sigma_{ij} \tag{7.1c}$$

where n_i represents the components of unit normal vector **n** of the surface; T_i is the stress vector at the point associated with **n**; $\sigma_{ji,j}$ represents the first derivative of σ_{ij} with respect to x_j; and F_i is the body force intensity. Any set of stresses σ_{ij}, body forces F_i, and external surface forces T_i that satisfies Eqs. (7.1a-c) is a statically admissible set.

Equations (7.1b and c) may be written in (*x,y,z*) notation as

$$\begin{aligned}
\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} + F_x &= 0 \\
\frac{\partial \tau_{yx}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} + F_y &= 0 \\
\frac{\partial \tau_{zx}}{\partial x} + \frac{\partial \tau_{zy}}{\partial y} + \frac{\partial \sigma_z}{\partial z} + F_z &= 0
\end{aligned} \tag{7.1d}$$

and

$$\tau_{xy} = \tau_{yx}, \quad \text{etc.} \tag{7.1e}$$

where σ_x, σ_y, and σ_z are the normal stress in (*x,y,z*) direction respectively; τ_{xy}, τ_{yx}, and so on, are the corresponding shear stresses in (*x,y,z*) notation; and F_x, F_y, and F_z are the body forces in (*x,y,z*) direction, respectively.

The principle of virtual work has proved a very powerful technique of solving problems and providing proofs for general theorems in solid mechanics. The equation of virtual work uses two

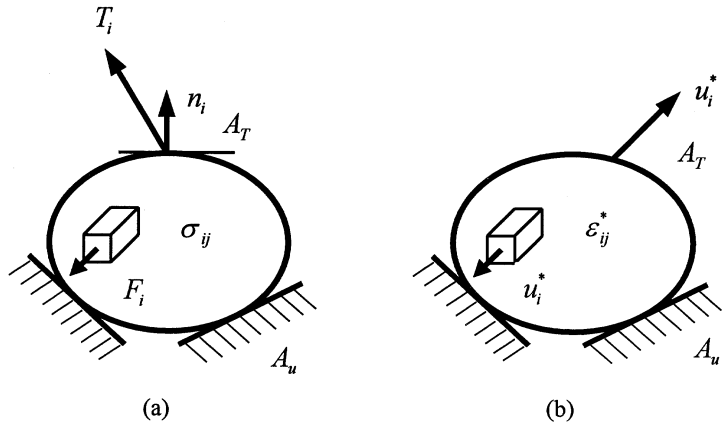


FIGURE 7.3 Two independent sets in the equation of virtual work.

independent sets of *equilibrium* and *compatible* (see Figure 7.3, where A_u and A_T represent displacement and stress boundary, respectively), as follows:

$$\underbrace{\int_A T_i u_i^* dA + \int_V F_i u_i^* dV}_{\text{compatible set}} = \underbrace{\int_V \sigma_{ij}\varepsilon_{ij}^* dV}_{\text{equilibrium set}} \tag{7.2}$$

or

$$\delta W_{\text{ext}} = \delta W_{\text{int}} \tag{7.3}$$

which states that the *external* virtual work (δW_{ext}) equals the *internal* virtual work (δW_{int}).

Here the integration is over the whole area A, or volume V, of the body. The stress field σ_{ij}, body forces F_i, and external surface forces T_i are a statically admissible set that satisfies Eqs. (7.1a–c). Similarly, the strain field ε_{ij}^* and the displacement u_i^* are a compatible kinematics set that satisfies displacement boundary conditions and Eq. (7.16) (see Section 7.3.1). This means the principle of virtual work applies only to small strain or small deformation.

The important point to keep in mind is that, neither the admissible equilibrium set σ_{ij}, F_i, and T_i (Figure 7.3a) nor the compatible set ε_{ij}^* and u_i^* (Figure 7.3b) need be the actual state, nor need the equilibrium and compatible sets be related to each other in any way. In the other words, these two sets are completely independent of each other.

7.2.2 Equilibrium Equation for Elements

For an infinitesimal material element, equilibrium equations have been summarized in Section 7.2.1, which will transfer into specific expressions in different methods. As in ordinary FEM or the displacement method, it will result in the following element equilibrium equations:

$$\{\bar{F}\}^e = [\bar{k}]^e \{\bar{d}\}^e \tag{7.4}$$

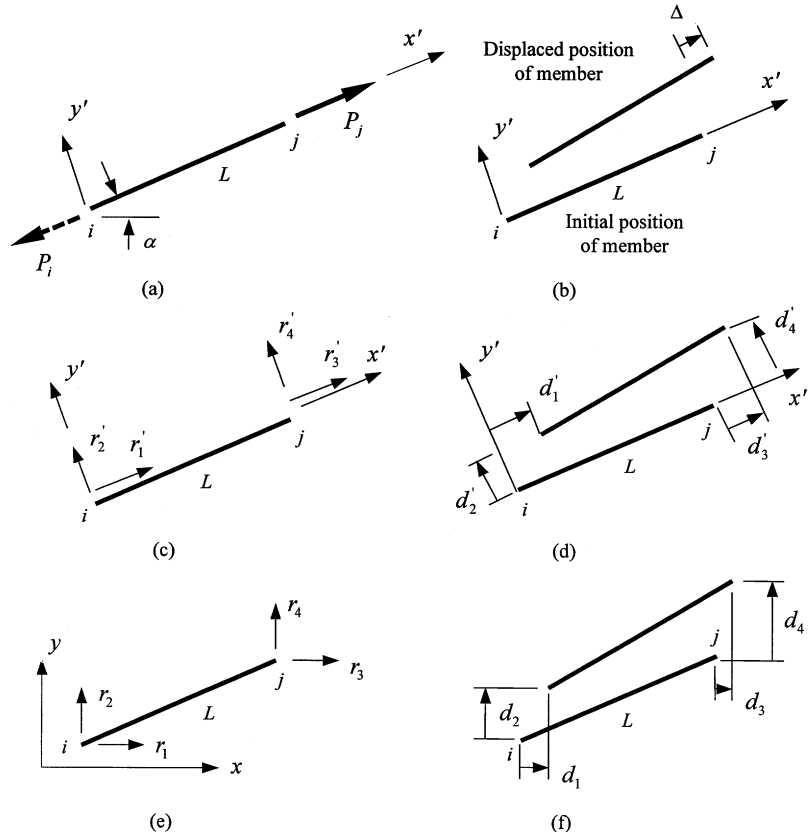


FIGURE 7.4 Plane truss member–end forces and displacements. (*Source*: Meyers, V.J., *Matrix Analysis of Structures,* New York: Harper & Row, 1983. With permission.)

where $\{\bar{F}\}^e$ and $\{\bar{d}\}^e$ are the element nodal force vector and displacement vector, respectively, while $[\bar{k}]^e$ is element stiffness matrix; the overbar here means in local coordinate system.

In the force method of structural analysis, which also adopts the idea of discretization, it is proved possible to identify a basic set of independent forces associated with each member, in that not only are these forces independent of one another, but also all other forces in that member are directly dependent on this set. Thus, this set of forces constitutes the minimum set that is capable of completely defining the stressed state of the member. The relationship between basic and local forces may be obtained by enforcing overall equilibrium on one member, which gives

$$\{\bar{F}\}^e = [L]\{P\}^e \tag{7.5}$$

where $[L]$ = the element force transformation matrix and $\{P\}^e$ = the element primary forces vector. It is important to emphasize that the physical basis of Eq. (7.5) is member overall equilibrium.

Take a conventional plane truss member for exemplification (see Figure 7.4), one has

$$\{\bar{k}\}^e = \begin{bmatrix} EA/l & 0 & -EA/l & 0 \\ 0 & 0 & 0 & 0 \\ -EA/l & 0 & EA/l & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{7.6}$$

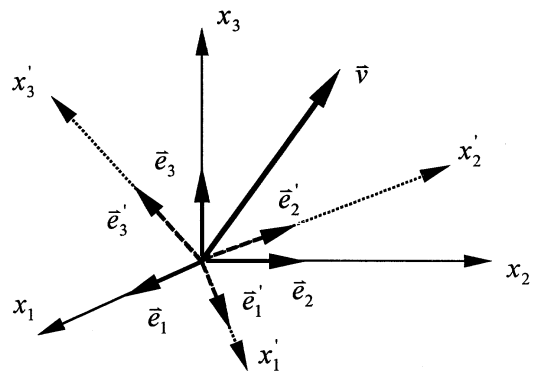


FIGURE 7.5 Coordinate transformation.

and

$$\begin{aligned}\{\bar{F}\}^e &= \{r_1' \quad r_2' \quad r_3' \quad r_4'\}^T \\ \{\bar{d}\}^e &= \{d_1' \quad d_2' \quad d_3' \quad d_4'\}^T \\ [L] &= \{-1 \quad 0 \quad 1 \quad 0\}^T \\ \{P\}^e &= \{P\}\end{aligned} \tag{7.7}$$

where EA/l = axial stiffness of the truss member and P = axial force of the truss member.

7.2.3 Coordinate Transformation

The values of the components of vector **V**, designated by v_1, v_2, and v_3 or simply v_i, are associated with the chosen set coordinate axes. Often it is necessary to reorient the reference axes and evaluate new values for the components of **V** in the new coordinate system. Assuming that **V** has components v_i and v_i' in two sets of right-handed Cartesian coordinate systems x_i (old) and x_i' (new) having the same origin (see Figure 7.5), and $\bar{e}_i$, $\bar{e}_i'$ are the unit vectors of x_i and x_i', respectively. Then

$$v_i' = l_{ij} v_j \tag{7.8}$$

where $l_{ji} = \bar{e}_j' \cdot \bar{e}_i = \cos(x_j', x_i)$, that is, the cosines of the angles between x_i' and x_j axes for i and j ranging from 1 to 3; and $[\alpha] = (l_{ij})_{3\times 3}$ is called coordinate transformation matrix from the old system to the new system.

It should be noted that the elements of l_{ij} or matrix $[\alpha]$ are not symmetrical, $l_{ij} \neq l_{ji}$. For example, l_{12} is the cosine of angle from x_1' to x_2 and l_{21} is that from x_2' to x_1 (see Figure 7.5). The angle is assumed to be measured from the primed system to the unprimed system.

For a plane truss member (see Figure 7.4), the transformation matrix from local coordinate system to global coordinate system may be expressed as

$$[\alpha] = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 & 0 \\ \sin\alpha & \cos\alpha & 0 & 0 \\ 0 & 0 & \cos\alpha & -\sin\alpha \\ 0 & 0 & \sin\alpha & \cos\alpha \end{bmatrix} \tag{7.9}$$

where α is the inclined angle of the truss member which is assumed to be measured from the global to the local coordinate system.

7.2.4 Equilibrium Equation for Structures

For discretized structure, the equilibrium of the whole structure is essentially the equilibrium of each joint. After assemblage,

For ordinary FEM or displacement method

$$\{F\} = [K]\{D\} \tag{7.10}$$

For force method

$$\{F\} = [A]\{P\} \tag{7.11}$$

where $\{F\}$ = nodal loading vector; $[K]$ = total stiffness matrix; $\{D\}$ = nodal displacement vector; $[A]$ = total forces transformation matrix; $\{P\}$ = total primary internal forces vector.

It should be noted that the coordinate transformation for each element from local coordinates to the global coordinate system must be done before assembly.

In the force method, Eq. (7.11) will be adopted to solve for internal forces of a statically determinate structure. The number of basic unknown forces is equal to the number of equilibrium equations available to solve for them and the equations are linearly independent. For statically unstable structures, analysis must consider their dynamic behavior. When the number of basic unknown forces exceeds the number of equilibrium equations, the structure is said to be statically indeterminate. In this case, some of the basic unknown forces are not required to maintain structural equilibrium. These are "extra" or "redundant" forces. To obtain a solution for the full set of basic unknown forces, it is necessary to augment the set of independent equilibrium equations with elastic behavior of the structure, namely, the force–displacement relations of the structure. Having solved for the full set of basic forces, we can determine the displacements by backsubstitution.

7.2.5 Influence Lines and Surfaces

In the design and analysis of bridge structures , it is necessary to study the effects intrigued by loads placed in various positions. This can be done conveniently by means of diagrams showing the effect of moving a unit load across the structures. Such diagrams are commonly called influence lines (for framed structures) or influence surfaces (for plates). Observe that whereas a moment or shear diagram shows the variation in moment or shear along the structure due to some particular position of load, an influence line or surface for moment or shear shows the variation of moment or shear at a *particular* section due to a unit load placed anywhere along the structure.

Exact influence lines for statically determinate structures can be obtained analytically by statics alone. From Eq. (7.11), the total primary internal forces vector $\{P\}$ can be expressed as

$$\{P\} = [A]^{-1}\{F\} \tag{7.12}$$

by which given a unit load at one node, the excited internal forces of all members will be obtained, and thus Eq. (7.12) gives the analytical expression of influence lines of all member internal forces for discretized structures subjected to moving nodal loads.

For statically indeterminate structures, influence values can be determined directly from a consideration of the geometry of the deflected load line resulting from imposing a unit deformation corresponding to the function under study, based on the principle of virtual work. This may better be demonstrated by a two-span continuous beam shown in Figure 7.6, where the influence line of internal bending moment M_B at section B is required.

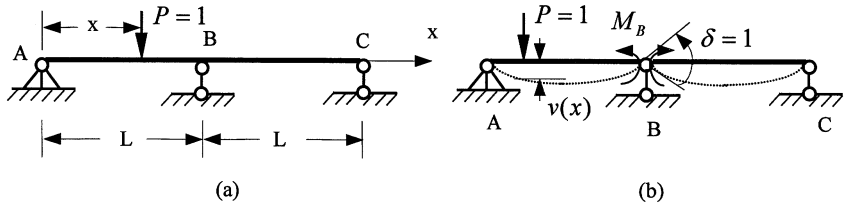


FIGURE 7.6 Influence line of a two-span continuous beam.

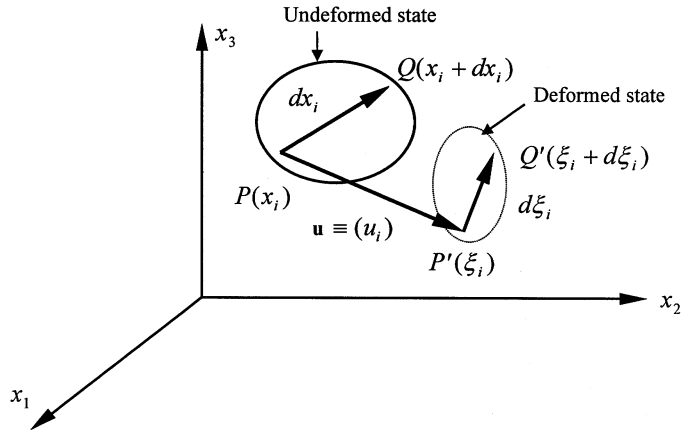


FIGURE 7.7 Deformation of a line element for Lagrangian and Eluerian variables.

Cutting section B to expose M_B and give it a unit relative rotation $\delta = 1$ (see Figure 7.6) and employing the principle of virtual work gives

$$M_B \cdot \delta = -P \cdot v(x) \tag{7.13}$$

Therefore,

$$M_B = -v(x) \tag{7.14}$$

which means the influence value of M_B equals to the deflection $v(x)$ of the beam subjected to a unit rotation at joint B (represented by dashed line in Figure 7.6b). Solving for $v(x)$ can be carried out easily referring to material mechanics.

7.3 Compatibility Equations

7.3.1 Large Deformation and Large Strain

Strain analysis is concerned with the study of deformation of a continuous body which is unrelated to properties of the body material. In general, there are two methods of describing the deformation of a continuous body, Lagrangian and Eulerian. The Lagrangian method employs the coordinates of each particle in the initial position as the independent variables. The Eulerian method defines the independent variables as the coordinates of each material particle at the time of interest.

Let the coordinates of material particle P in a body in the initial position be denoted by x_i (x_1, x_2, x_3) referred to the fixed axes x_i, as shown in Figure 7.7. And the coordinates of the particle after deformation are denoted by ξ_i (ξ_1, ξ_2, ξ_3) with respect to axes x_i. As for the independent variables, Lagrangian formulation uses the coordinates (x_i) while Eulerian formulation employs the coordinates (ξ_i). From motion analysis of line element PQ (see Figure 7.7), one has

For Lagrangian formulation, the Lagrangian strain tensor is

$$\varepsilon_{ij} = \frac{1}{2}\,(u_{i,j} + u_{j,i} + u_{r,i}u_{r,j}) \tag{7.15}$$

where $u_{i,j} = \partial u_i / \partial x_j$ and all quantities are expressed in terms of (x_i).

For Eulerian formulation, the Eulerian strain tensor is

$$E_{ij} = \frac{1}{2}\,(u_{i/j} + u_{j/i} + u_{r/i}u_{r/j}) \tag{7.16}$$

where $u_{i/j} = \partial u_i / \partial \xi_j$ and all quantities are described in terms of (ξ_i).

If the displacement derivatives $u_{i,j}$ and $u_{i/j}$ are not so small that their nonlinear terms cannot be neglected, it is called large deformation, and the solving of u_i will be rather difficult since the nonlinear terms appear in the governing equations.

If both the displacements and their derivatives are small, it is immaterial whether the derivatives in Eqs. (7.15) and (7.16) are calculated using the (x_i) or the (ξ_i) variables. In this case both Lagrangian and Eulerian descriptions yield the same strain–displacement relationship:

$$\varepsilon_{ij} = E_{ij} = \frac{1}{2}\,(u_{i,j} + u_{j,i}) \tag{7.17}$$

which means small deformation, the most common in structural engineering.

For given displacements (u_i) in strain analysis, the strain components (ε_{ij}) can be determined from Eq. (7.17). For prescribed strain components (ε_{ij}), some restrictions must be imposed on it in order to have single-valued continuous displacement functions u_i, since there are six equations for three unknown functions. Such restrictions are called compatibility conditions, which for a simply connected region may be written as

$$\varepsilon_{ij,kl} + \varepsilon_{kl,ij} - \varepsilon_{ik,jl} - \varepsilon_{ji,ik} = 0 \tag{7.18a}$$

or, expanding these expressions in the (*x, y, z*) notations, it gives

$$\begin{gathered}
\frac{\partial^2 \varepsilon_x}{\partial y^2} + \frac{\partial^2 \varepsilon_y}{\partial x^2} = 2\frac{\partial^2 \varepsilon_{xy}}{\partial x \partial y} \\
\frac{\partial^2 \varepsilon_y}{\partial z^2} + \frac{\partial^2 \varepsilon_z}{\partial y^2} = 2\frac{\partial^2 \varepsilon_{yz}}{\partial y \partial z} \\
\frac{\partial^2 \varepsilon_z}{\partial x^2} + \frac{\partial^2 \varepsilon_x}{\partial z^2} = 2\frac{\partial^2 \varepsilon_{zx}}{\partial z \partial x} \\
\frac{\partial}{\partial x}\left(-\frac{\partial \varepsilon_{yz}}{\partial x} + \frac{\partial \varepsilon_{zx}}{\partial y} + \frac{\partial \varepsilon_{xy}}{\partial z}\right) = \frac{\partial^2 \varepsilon_x}{\partial y \partial z} \\
\frac{\partial}{\partial y}\left(-\frac{\partial \varepsilon_{zx}}{\partial y} + \frac{\partial \varepsilon_{xy}}{\partial z} + \frac{\partial \varepsilon_{yz}}{\partial x}\right) = \frac{\partial^2 \varepsilon_y}{\partial z \partial x} \\
\frac{\partial}{\partial z}\left(-\frac{\partial \varepsilon_{xy}}{\partial z} + \frac{\partial \varepsilon_{yz}}{\partial x} + \frac{\partial \varepsilon_{zx}}{\partial y}\right) = \frac{\partial^2 \varepsilon_z}{\partial x \partial y}
\end{gathered} \tag{7.18b}$$

Any set of strains ε_{ij} and displacements u_i, that satisfies Eqs. (7.17) and (7.18a) or (7.18b), as well as displacement boundary conditions, is a kinematics admissible set, or a compatible set.

7.3.2 Compatibility Equation for Elements

For ordinary FEM, compatibility requirements are self-satisfied in the formulating procedure. As for equilibrium equations, a basic set of independent displacements can be identified for each member, and the kinematics relationships between member basic displacements and member–end displacements of one member can be given as follows:

$$\{\Delta\}^e = [L]^T \{\bar{d}\}^e \tag{7.19}$$

where $\{\Delta\}^e$ is element primary displacement vector, $[L]$ and $\{\bar{d}\}^e$ have been shown in Section 7.2.2. For plane truss member, $\{\Delta\}^e = \{\Delta\}$, where Δ is the relative displacement of the member (see Figure 7.5). It should also be noted that the physical basis of Eq. (7.19) is the overall compatibility of the element.

7.3.3 Compatibility Equation for Structures

For the whole structure, one has the following equation after assembly process:

$$\{\Delta\} = [A]^T \{D\} \tag{7.20}$$

where $\{\Delta\}$ = total primary displacement vector; $\{D\}$ = total nodal displacement vector; and $[A]^T$ = the transposition of $[A]$ described in Section 7.2.4.

A statically determinate structure is kinematically determinate. Given a set of basic member displacements, there are a sufficient number of compatibility relationships available to allow the structure nodal displacements to be determined. In addition to their application to settlement and fabrication error loading, thermal loads can also be considered for statically determinate structures. External forces on a structure cause member distortions and, hence, nodal displacements, but before such problems can be solved, the relationships between member forces and member distortions must be developed. These will be shown in Section 7.5.1.

7.3.4 Contragredient Law

During the development of the equilibrium and compatibility relationships, it has been noticed that various corresponding force and displacement transformations are the transposition of each other, as shown not only in Eqs. (7.5) and (7.19) of element equilibrium and compatibility relations, but also in Eqs. (7.11) and (7.20) of global equilibrium and compatibility relations, although each pair of these transformations was obtained independently of the other in the development. These special sets of relations are termed the contragredient law which was established on the basis of virtual work concepts. Therefore, after a particular force transformation matrix is obtained, the corresponding displacement transformation matrix would be immediately apparent, and it remains valid to the contrary.

7.4 Constitutive Equations

7.4.1 Elasticity and Plasticity

A material body will produce deformation when subjected to external excitations. If upon the release of applied actions the body recovers its original shape and size, it is called an *elastic* material, or

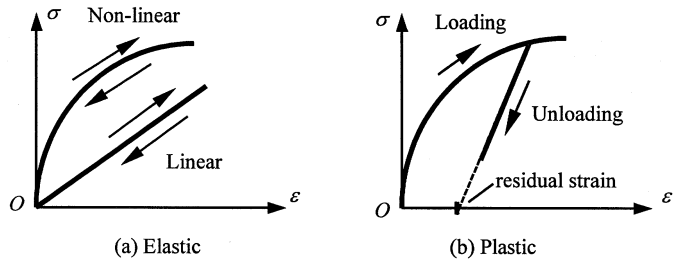


FIGURE 7.8 Sketches of behavior of elastic and plastic materials.

one can say the material has the characteristic of *elasticity*. Otherwise, it is a *plastic* material or a material with *plasticity*. For an elastic body, the current state of stress depends only on the current state of deformation; that is, the constitutive equations for elastic material are given by

$$\sigma_{ij} = F_{ij}(\varepsilon_{kl}) \tag{7.21}$$

where F_{ij} is called the elastic response function. Thus, the elastic material behavior described by Eq. (7.21) is reversible and path independent (see Figure 7.8a), in which case the material is usually termed *Cauchy elastic* material.

Reversibility and path independence are not exhibited by plastic materials (see Figure 7.8b). In general, a plastic material does not return to its original shape; *residual* deformation and stresses remain inside the body even when all external tractions are removed. As a result, it is necessary for plasticity to extend the elastic stress–strain relations into the plastic range where permanent plastic stain is possible. It makes the solution of a solid mechanics problem more complicated.

7.4.2 Linear Elastic and Nonlinear Elastic Behavior

Just as the term *linear* implies, linear elasticity means the elastic response function F_{ij} of Eq. (7.21) is a linear function, whose most general form for a Cauchy elastic material is given by

$$\sigma_{ij} = B_{ij} + C_{ijkl}\varepsilon_{kl} \tag{7.22}$$

where B_{ij} = components of initial stress tensor corresponding to the *initial strain-free* state (i.e., $\varepsilon_{ij} = 0$), and C_{ijkl} = tensor of material elastic constants.

If it is assumed that $B_{ij} = 0$, Eq. (7.22) will be reduced to

$$\sigma_{ij} = C_{ijkl}\varepsilon_{kl} \tag{7.23}$$

which is often referred to as the generalized Hook's law.

For an *isotropic* linear elastic material, the elastic constants in Eq. (7.23) must be the same for all directions and thus C_{ijkl} must be an isotropic fourth-order tensor, which means that there are only two independent material constants. In this case, Eq. (7.23) will reduce to

$$\sigma_{ij} = \lambda\varepsilon_{kk}\delta_{ij} + 2\mu\varepsilon_{ij} \tag{7.24}$$

where λ and μ are the two material constants, usually called *Lame's constants*; δ_{ij} = *Kronecker delta* and ε_{kk} = the summation of the diagonal terms of ε_{ij} according to the *summation convention*, which means that, whenever a subscript occurs twice in the same term, it is understood that the subscript is to be summed from 1 to 3.

If the elastic response function F_{ij} in Eq. (7.21) is not linear, it is called nonlinear elastic, and the material exhibits nonlinear mechanical behavior even when sustaining small deformation. That is, the material elastic "constants" do not remain constant any more, whereas the deformation can still be reversed completely.

7.4.3 Geometric Nonlinearity

Based on the sources from which it arises, nonlinearity can be categorized into material nonlinearity (including nonlinear elasticity and plasticity) and geometric nonlinearity. When the nonlinear terms in the strain–displacement relations cannot be neglected (see Section 7.3.1) or the deflections are large enough to cause significant changes in the structural geometry, it is termed geometric nonlinearity. It is also called large deformation, and the principle of superposition derived from small deformations is no longer valid. It should be noted that for accumulated large displacements with small deformations, it could be linearized by a step-by-step procedure.

According to the different choice of reference frame, there are two types of Lagrangian formulation: the total Lagrangian formulation, which takes the original unstrained configuration as the reference frame, and the updated Lagrangian formulation based on the latest-obtained configuration, which are usually carried out step by step. Whatever formulation one chooses, a geometric stiffness matrix or initial stress matrix will be introduced into the equations of equilibrium to take account of the effects of the initial stresses on the stiffness of the structure. These depend on the magnitude or conditions of loading and deformations, and thus cause the geometric nonlinearity. In beam–column theory, this is well known as the second-order or the P–Δ effect. For detailed discussions, see Chapter 36.

7.5 Displacement Method

7.5.1 Stiffness matrix for elements

In displacement method, displacement components are taken as primary unknowns. From Eqs. (7.5) and (7.19) the equilibrium and compatibility requirements on elements have been acquired. For a statically determinate structure, no subsidiary conditions are needed to obtain internal forces under nodal loading or the displaced position of the structure given the basic distortion such as support settlement or fabrication errors. For a statically indeterminate structure, however, supplementary conditions, namely, the constitutive law of materials constructing the structure, should be incorporated for the solution of internal forces as well as nodal displacements.

From structural mechanics, the basic stiffness relationships for a member between basic internal forces and basic member–end displacements can be expressed as

$$\{P\}^e = [k]^e \{\Delta\}^e \tag{7.25}$$

where $[k]^e$ is the element basic stiffness matrix, which can be termed $[\,EA/l\,]$ for a conventional plane truss member (see Figure 7.4).

Substitution of Eqs. (7.19) and (7.25) into Eq. (7.5) yields

$$\begin{aligned} \left\{\bar{F}\right\}^e &= [L][k]^e [L]^T \left\{\bar{d}\right\}^e \\ &= \left[\bar{k}\right]^e \left\{\bar{d}\right\}^e \end{aligned} \tag{7.26}$$

where

$$\left[\bar{k}\right]^e = [L][k]^e [L]^T \tag{7.27}$$

is called the element stiffness matrix, the same as in Eq. (7.4). It should be kept in mind that the element stiffness matrix $\left[\bar{k}\right]^e$ is symmetric and singular, since given the member–end forces, member–end displacements cannot be determined uniquely because the member may undergo rigid body movement.

7.5.2 Stiffness Matrix for Structures

Our final aim is to obtain equations that define approximately the behavior of the whole body or structure. Once the element stiffness relations of Eq. (7.26) is established for a generic element, the global equations can be constructed by an assembling process based on the law of compatibility and equilibrium, which are generally expressed in matrix notation as

$$\{F\} = [K]\{D\} \tag{7.28}$$

where $[K]$ is the stiffness matrix for the whole structure. It should be noted that the basic idea of assembly involves a minimization of *total* potential energy, and the assembled stiffness matrix $[K]$ is *symmetric* and *banded* or *sparsely populated.*

Eq. (7.28) tells us the capabilities of a structure to withstand applied loading rather than the true behavior of the structure if boundary conditions are not introduced. In other words, without boundary conditions, there can be an infinite number of possible solutions since stiffness matrix $[K]$ is singular; that is, its determinant vanishes. Hence, Eqs. (7.28) should be modified to reflect boundary conditions and the final modified equations are expressed by inserting overbars as

$$\left\{\bar{F}\right\} = \left[\bar{K}\right]\left\{\bar{D}\right\} \tag{7.29}$$

7.5.3 Matrix Inversion

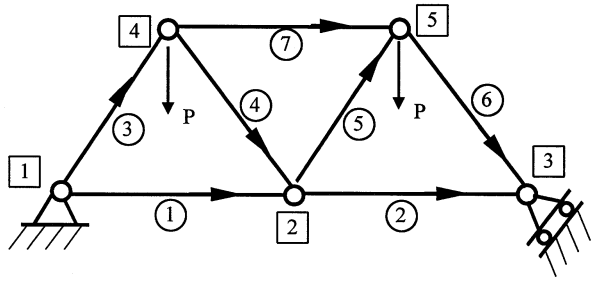
It has been shown that sets of simultaneous algebraic equations are generated in the application of both the displacement method and the force method in structural analysis, which are usually linear. The coefficients of the equations are constant and do not depend on the magnitude or conditions of loading and deformations, since linear Hook's law is generally assumed valid and small strains and deformations are used in the formulation. Solving Eq. (7.29) is, namely, to invert the modified stiffness matrix $\left[\bar{K}\right]$. This requires tremendous computational efforts for large-scale problems. The equations can be solved by using direct, iterative, or other methods. Two steps of elimination and backsubstitution are involved in the direct procedures, among which are Gaussian elimination and a number of its modifications. These are some of the most widely used sets of direct methods because of their better accuracy and small number of arithmetic operations.

7.5.4 Special Consideration

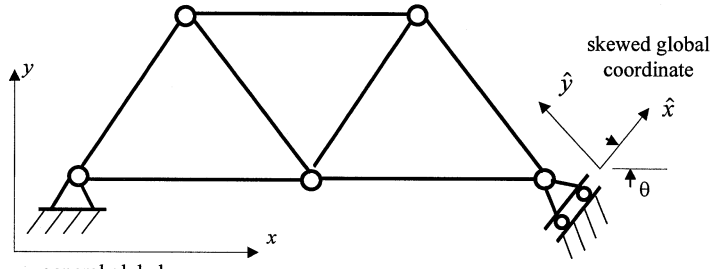
In practice, a variety of special circumstances, ranging from loading to internal member conditions and supporting conditions, should be given due consideration in structural analysis.

Initially strains, which are not directly associated with stresses, result from two causes, thermal loading or fabrication error. If the member with initial strains is unconstrained, there will be a set of initial member–end displacements associated with these initial strains, but nevertheless no initial member–end forces. For a member constrained to act as part of a structure, the general member force–displacement relationships will be modified as follows:

$$\left\{\bar{F}\right\}^e = \left[\bar{k}\right]^e\left(\left\{\bar{d}\right\}^e - \left\{\bar{d_0}\right\}^e\right) \tag{7.30a}$$



(a) Dimensions and Loading



(b) Mixed coordinate model

FIGURE 7.9 Plane truss with skewed support.

or

$$\{\bar{F}\}^e = [\bar{k}]^e\{\bar{d}\}^e + \{R_{F0}\}^e \tag{7.30b}$$

where

$$\{R_{F0}\}^e = -[\bar{k}]^e\{\bar{d}_0\}^e \tag{7.31}$$

are fixed-end forces, and $\{\bar{d}_0\}^e$ a vector of initial member–end displacements for the member.

It is interesting to note that a support settlement may be regarded as an initial strain. Moreover, initial strains including thermal loading and fabrication errors, as well as support settlements, can all be treated as external excitations. Hence, the corresponding fixed-end forces as well as the equivalent nodal loading can be obtained which makes the conventional procedure described previously still practicable.

For a skewed support which provides a constraint to the structure in a nonglobal direction, the effect can be given due consideration by adapting a skewed global coordinate (see Figure 7.9) by introducing a skewed coordinate at the skewed support. This can perhaps be better demonstrated by considering a specific example of a plane truss shown in Figure 7.9. For members jointed at a skewed support, the coordinate transformation matrix will takes the form of

$$[\alpha] = \begin{bmatrix} \cos\alpha_i & -\sin\alpha_i & 0 & 0 \\ \sin\alpha_i & \cos\alpha_i & 0 & 0 \\ 0 & 0 & \cos\alpha_j & -\sin\alpha_j \\ 0 & 0 & \sin\alpha_j & \cos\alpha_j \end{bmatrix} \tag{7.32}$$

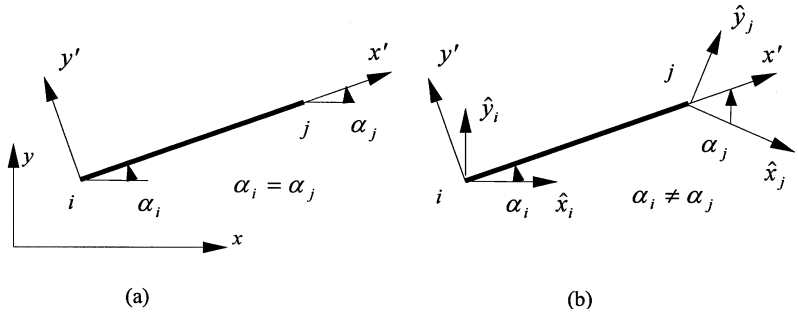


FIGURE 7.10 Plane truss member coordinate transformation. (a) Normal global coordinate; (b) skewed global coordinate.

where α_i and α_j are inclined angles of truss member in skewed global coordinate (see Figure 7.10), say, for member 2 in Figure 7.9, $\alpha_i = 0$ and $\alpha_j = -\theta$.

For other special members such as inextensional or variable cross section ones, it may be necessary or convenient to employ special member force–displacement relations in structural analysis. Although the development and programming of a stiffness method general enough to take into account all these special considerations is formidable, more important perhaps is that the application of the method remains little changed. For more details, readers are referred to Reference. [5].

7.6 Substructuring and Symmetry Consideration

For highly complex or large-scale structures, one is required to solve a very large set of simultaneous equations, which are sometimes restricted by the computation resources available. In that case, special data-handling schemes like static condensation are needed to reduce the number of unknowns by appropriately numbering nodal displacement components and disposition of element force–displacement relations. Static condensation is useful in dynamic analysis of framed structures since the rotatory moment of inertia is usually neglected.

Another scheme physically partitions the structure into a collection of smaller structures called "substructures," which can be processed by parallel computers. In static analysis, the first step of substructuring is to introduce imaginary fixed inner boundaries, and then release all inner boundaries simultaneously, which gives rise to a subsequent analysis of these substructure series in a smaller scale. It is essentially the patitioning of Eq. (7.28) as follows. For the rth substructure, one has

Case (α) : *Introducing inner fixed boundaries*

$$\begin{bmatrix} K_{bb} & K_{bi} \\ K_{ib} & K_{ii} \end{bmatrix}^{(r)} \begin{Bmatrix} 0 \\ D_i^{\alpha} \end{Bmatrix}^{(r)} = \begin{Bmatrix} F_b^{\alpha} \\ F_i \end{Bmatrix}^{(r)} \tag{7.33}$$

Case (β) : *Releasing all inner fixed boundaries*

$$\begin{bmatrix} K_{bb} & K_{bi} \\ K_{ib} & K_{ii} \end{bmatrix}^{(r)} \begin{Bmatrix} D_b \\ D_i^{\beta} \end{Bmatrix}^{(r)} = \begin{Bmatrix} F_b^{\beta} \\ 0 \end{Bmatrix}^{(r)} \tag{7.34}$$

where subscripts b and i denote inner fixed and free nodes, respectively.

Combining Eqs. (7.33) and (7.34) gives the force–displacement relations for enlarged elements — substructures which may be expressed as

$$\left[K_b\right]^{(r)}\left\{D_b\right\}^{(r)} = \left\{F_b\right\}^{(r)} \tag{7.35}$$

which is analogous to Eq. (7.26) and $\left\{F_b\right\}^{(r)} = \left\{F_b^{(r)}\right\} - \left[K_{bi}^{(r)}\right]\left[K_{ii}^{(r)}\right]^{-1}\left\{F_i^{(r)}\right\}$. And thereby the conventional procedure is still valid.

Similarly, in the cases of structural symmetry of geometry and material, proper consideration of loading symmetry and antisymmetry can give rise to a much smaller set of governing equations.

For more details, please refer to the literature on structural analysis.

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