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# Design of steel frames without consideration of effective length

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In this paper, limits that have been suggested in the literature for use of $K=1$ and/or for neglecting $P\Delta$ moments in design are reviewed within the context of AISC LRFD practice. The paper then presents general equations that give the error in the AISC LRFD beam-column interaction equations associated with the use of $K=1$. In the development of these error equations, it is assumed that the design is based on second-order elastic forces calculated as per the requirements of the present AISC LRFD Specification, i.e., by use of approximate amplifiers or by direct analysis, with geometric imperfection or notional load effects not included in the analysis calculations. The influence of key variables on the error is studied, and recommendations are provided for when the design of steel frames by AISC LRFD may be based on $K=1$. The paper closes by comparing design strengths with and without effective length to the results from elastic-plastic hinge and rigorous plastic zone analyses for several 'maximum error' examples. This provides an assessment of the accuracy of upper-bound error estimates, and of the implications of using $K=1$ relative to the theoretical inelastic frame behaviour. The discussions and recommendations are applicable for any type of steel frame (i.e., frames with fully or partially restrained connections, and unbraced or partially braced frames) in which a storey-by-storey sidesway stability assessment is appropriate. © 1997 Elsevier Science Ltd.

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## 1. Introduction

There has been much discussion during recent decades about the appropriate calculation of effective length factors in sway frames, as well as whether or not it is necessary to consider effective length at all when the engineer has conducted a second-order elastic analysis to determine the design forces[1,2]. Therefore, it is no surprise that present specifications and standards reflect varying philosophies on this issue. the AISC LRFD Specification[3] requires the calculation of column buckling loads, which is often handled implicitly by determination of effective lengths, whereas many of the other design standards[4–6], do not require any buckling analysis or effective length calculation. These standards use 'notional horizontal load' or 'equivalent imperfection' approaches in computing second-order elastic forces to account for the effects of geometric imperfections and distributed plasticity on a frame's sidesway stability. Conversely, the AISC LRFD Specification does not require any consideration of geometric imperfections or distributed plasticity within the design analysis, but accounts for these effects through the column strength equations (by basing the column strengths on buckling effective lengths).

The ASCE Technical Committee on LRFD is due to publish a report[7] that identifies and contrasts several appropriate effective length and notional load approaches within the context of AISC LRFD practice. This report emphasizes that the use of $K=1$ in the design of steel frames can result in significant unconservative errors, even if second-order elastic forces are computed, unless an alternative device such as the notional horizontal load concept is employed with all the load combinations. However, in many situations, the error associated with disregarding the effective length or notional load approximations is small. Higgins[8] confirmed this fact in 1964, shortly after the effective length concept was first introduced within the AISC Specifications, by the disclaimer

> 'It should not be construed from the following remarks that, from now on, all tier buildings, or even very many, must be designed on the basis of overall instability unless provided with an extensive bracing system.'

Significant design effort is often required to obtain appropriate buckling solutions and/or the associated effective length factors. Also, the proper application of notional loads is tedious in certain cases. Therefore, it is desirable to define simple checks that can be used to ascertain when the unconservative error associated with using $K = 1$ is negligible, without the need for alternative devices such as notional load. The major goals of this paper are to clarify the errors associated with this simplification within the context of the AISC LRFD Specification[9], and to suggest general rules for when the design of frames can be based on $K = 1$ in current LRFD practice.

## Notation

| | |
|---|---|
| $A_g$ | gross cross-section area |
| $B_2$ | Storey sidesway amplification factor at factored load levels |
| $B_{2s}$ | Storey sidesway amplification factor at service load levels |
| $C_L$ | $P\delta$ stiffness reduction factor for a column member |
| $(C_L)_{avg}$ | weighted average value of $C_L$ over storey |
| $E$ | elastic modulus |
| $F_{a(L)}$ | allowable stress for column design in AISC-ASD[3], based on actual column length $L$ |
| $F_y$ | yield stress |
| $G$ | $\Sigma(I_c/L_c)/\Sigma(I_g/L'_g)$ at a beam-column joint |
| $H$ | column shear force obtained from a lateral load analysis |
| $I$ | cross-section moment of inertia |
| $I_c$ | moment of inertia of column |
| $I_g$ | moment of inertia of girder |
| $K$ | effective length factor |
| $K_{C_L}$ | storey-based effective length factor, with $P\delta$ effects accounted for through $C_L$ |
| $K_x$, $K_y$ | storey-based effective length factors for flexural buckling about $x$- and $y$-axes |
| $L$, $L_c$ | actual column length |
| $L_g$ | actual girder length |
| $L'_g$ | equivalent girder length for calculation of column effective lengths by AISC LRFD sidesway-uninhibited alignment chart[9] |
| $L_x$, $L_y$ | column unsupported lengths for flexural buckling about $x$- and $y$-axes |
| $M_F$, $M_N$ | sidesway moments at far and near ends of a girder, relative to joint at which $G$ is being calculated, used for calculating $L'_g$ |
| $M_n$ | member nominal strength for bending within plane of frame |
| $M_p$ | plastic bending capacity in absence of axial compression |
| $M_u$ | member maximum second-order elastic moment for bending within plane of frame |
| $N$ | ratio of factored gravity load on all columns of a storey to that supported by columns of lateral-resisting system |
| $P$ | axial force |
| $P_{e(C_L)}$ | column axial force at incipient elastic buckling of storey, based on $K_{C_L}$ |
| $P_{e(R_L)}$ | column axial force at incipient elastic buckling of storey, based on approximation of $1/[1 + (C_L)_{avg}]$ by $(0.85 + 0.15\,R_L)$ |
| $P_n$ | nominal compressive strength based on an appropriate column effective length |
| $P_{n(L)}$ | nominal compressive strength based on actual column length |
| $P_L$ | column stiffness pertaining to a unit rotation of column chord, $\Delta/L = 1$ |
| $P_s$ | axial force at service load levels |
| $P_u$ | axial force at factored load levels |
| $P_y$ | member yield load, $P_y = A_g F_y$ |
| $R_L$ | ratio of loads on all leaning columns in a storey to total vertical load supported by storey |
| $S_L$ | approximate ratio of storey yield strength to elastic storey buckling capacity |
| $e$ | error in column axial strength associated with use of $K = 1$ compared to column strength based on an elastic effective length factor |
| $f_a$ | applied axial stress at service load conditions |
| $r$ | radius of gyration |
| $r_x$, $r_y$ | radius of gyration about $x$- and $y$-axes of cross-section |
| $\Delta$ | relative transverse displacement between ends of a member |
| $\Delta_{oh}$ | first-order drift of storey due to $\Sigma H_{nonleaner}$ |
| $\beta$ | sidesway stiffness coefficient, i.e., $P_L = (\beta EI)/L^2$ |
| $\epsilon$ | error in LRFD beam-column interaction equation associated with use of $K = 1$, when this equation equals 1.0 (i.e., corresponding interaction equation based on an elastic effective length factor attains a value equal to $(1 + \epsilon)$ when interaction equation based on $K = 1$ is equal to 1.0) |
| $\epsilon_{max}$ | maximum value of $\epsilon$ for a specified $B_2$ limit, associated with $C_L \leq 0.176$, $P_u/P_y \geq 0.05$, $L/r \geq 10$, and $F_y \geq 250\ \text{N/mm}^2$, approximated by equation (25) |
| $\delta$ | deflection of member axis from rotated member chord |
| $\lambda_c$ | column slenderness parameter based on effective length $KL$ |
| $\lambda_{c(L)}$ | column slenderness parameter based on actual column length $L$ |
| $\phi_b$, $\phi_c$ | resistance factors for bending and axial compression |

## 2. Prior recommendations and design provisions

### 2.1. SSRC recommendations and AISC provisions

Research by Lu *et al.*[10] and Liapunov[11] demonstrated that within the context of AISC Allowable Stress Design[3], certain classes of rigidly connected unbraced frames can be designed neglecting $P\Delta$ and $P\delta$ effects in the calculation of system forces, and with the column axial strength computed based on $K = 1$. Limits that define when these conditions exist were developed based on parametric studies of 17 frames. These studies included a representative sample of practical frame geometries and loadings. Several elastic designs of these frames were performed, with one set of designs based on first-order moments and $K = 1$. These designs were subjected to second-order elastic-plastic-hinge and/or distributed plasticity analyses to ascertain whether their ultimate capacities were sufficient. The limits proposed based on this research are[1]

- The ratios $f_a/F_{a(L)}$ and $f_a/0.6F_y$ are not to exceed 0.75, where $f_a$ is the applied axial stress at service load conditions, and $F_{a(L)}$ is the allowable stress in ASD based on $K = 1$ (or $KL = L$)
- The maximum in-plane column slenderness ratio $L/r$ is not to exceed 35
- The bare-frame first-order drift index, $\Delta_{oh}/L$, is to be limited such that

$$\frac{\Delta_{oh}}{L} \leq \frac{1}{7} \frac{\sum\limits_{nonleaner} H}{\sum\limits_{all} P_s} \tag{1}$$

where $L$ is the storey height, $\sum\limits_{nonleaner} H$ is the total storey shear due to service lateral loads, $\Delta_{oh}$ is the first-order drift of the storey due to $\sum\limits_{nonleaner} H$, and $\sum\limits_{all} P_s$ is the total service level gravity load on the storey (the symbols $\sum\limits_{nonleaner}$ and $\sum\limits_{all}$ indicate summation over only the lateral-resisting columns within the storey or summation over all the columns, respectively). As a result, the approximate sidesway amplification at service load levels

$$B_{2s} = \frac{1}{1 - \dfrac{\sum\limits_{all} P_s}{\sum\limits_{nonleaner} P_L}} = \frac{1}{1 - \dfrac{\sum\limits_{all} P_s}{\sum\limits_{nonleaner} (HL)/\Delta_{oh}}} \tag{2}$$

is limited to a maximum value of 1.17

The above three requirements are simply the maximum values encountered in the design studies, and it has been suggested[1] that these recommendations are only tentative. These limits are cited in the AISC ASD Commentary[3]. However, the equivalent recommendations are not included in the AISC LRFD Manual[9].

The above limits may be placed in the context of AISC LRFD as follows. As has been observed[7,12], if the traditional allowable stress column equations[3] are multiplied by $1.67\phi_c A_g$, the LRFD column design strength ($\phi_c P_n$) is closely approximated. In fact, the LRFD column equations were developed as an approximate fit to the ASD equations at a live to dead load ratio of 1.1 and $\lambda_c = 1$[13], where $\lambda_c$ is the column slenderness parameter $(KL/r)(1/\pi)\sqrt{F_y/E}$. Therefore, upon recognizing that $f_a/F_{a(L)} \leq 0.75$ is the only relevant limit in the context of LRFD[7], the first restriction in the above list may be expressed as

$$1.67\phi_c A_g f_a \leq 0.75\ [1.67\phi_c A_g F_{a(L)}] \tag{3a}$$

After substituting $\phi_c = 0.85$ and $A_g f_a = P_s$ on the left-hand side of this equation, and $1.67A_g F_{a(L)} = P_{n(L)}$ on the right-hand side, equation (3$a$) can be written as

$$\frac{1.42}{(P_u/P_s)} P_u \leq 0.75\ [\phi_c P_{n(L)}] \tag{3b}$$

where $P_u$ is the factored axial load for strength design, $P_s$ is the applied axial force at the corresponding service load level, and $P_{n(L)}$ is the nominal axial strength obtained from the LRFD column equations based on $K = 1$. If the maximum $L/r$ limit of 35 (the second restriction in the above list) is considered along with the LRFD load factors for strength design, and if A36 steel is assumed, the above equation translates to a limit on $P_u$ ranging from $0.55P_y$ for $P_u/P_s = 1.3$ to $0.63P_y$ for $P_u/P_s = 1.5$ (1.3 to 1.5 representing the range of $P_u/P_s$ for a large percentage of designs[13]).

Equation (1) is applicable only at the nominal (i.e., unfactored) load levels in LRFD. This equation translates to a maximum allowable sidesway amplification at factored load levels[9]

$$B_2 = \frac{1}{1 - \dfrac{\sum\limits_{all} P_u}{\sum\limits_{nonleaner} P_L}} = \frac{1}{1 - \dfrac{\sum\limits_{all} P_u}{\sum\limits_{nonleaner} \dfrac{HL}{\Delta_{oh}}}} \tag{4}$$

that generally would be greater than 1.17 (the limit on equation (2)), depending on the ratio $\sum\limits_{all} P_u / \sum\limits_{all} P_s$.

### 2.2. *Other recommendations*

Several current limit states standards give rules for when stability effects can be neglected that are more restrictive than the above limits. Eurocode 3[5] provides one of the most extensive discussions on this issue. It restricts the ratio of the total factored gravity load to the elastic buckling load for which the engineer is allowed to design a frame as 'non-sway' (if a frame is considered nonsway, the design may be based on $K \leq 1$ and first-order forces). This limit corresponds to the load level at which the gravity load is 1/10 of the associated elastic sidesway buckling value. An equation that is identical to an upper bound on $B_2$ of 1.11 is suggested for checking this limit. A similar restriction is stated in the Australian AS4100 limit states design standard for assessment of when second-order effects may be neglected in plastic design of building frames[6]. These limits may be traced back to recommendations by Wood[14] and Horne[15].

The 1991 NEHRP Recommended Provisions for the Development of Seismic Regulations for New Buildings[16] provide equations for when $P\Delta$ effects on the member forces and storey drifts may be neglected in seismic design. One note of particular interest regarding the NEHRP provisions is they suggest that when calculating the vertical load for purposes of determining $P\Delta$ forces for seismic design, the load factors need not exceed 1.0. As a result, the NEHRP equations translate to a requirement that $B_{2s}$ must be less than or equal to 1.11 for the $P$-$\Delta$ effects on the member forces and for storey drifts to be neglected (a somewhat more liberal limit than those of Eurocode 3[5] and AS4100[6]).

### 2.3. *Summary and observations*

To the knowledge of the authors, with the exception of the work by Lu *et al.* and Liapunov discussed in Reference 1, there are no published studies available that attempt to quantify the error associated with neglecting effective length in the design of steel frames. Furthermore, the studies summarized in Reference 1 are not comprehensive. The frames considered involved an extensive range of practical frame designs, but none of the frames supported leaning columns, and none were representative of the most severe stability-critical cases such as the Kanchanalai[17] frames utilized in the development of the AISC LRFD beam-column equations[7]. All the study frames were highly redundant, and had substantial inelastic redistribution of forces prior to reaching their limits of maximum resistance.

None of the recommendations cited above focus on the use of $K = 1$ in a design based on second-order elastic forces (calculated either by second-order analysis or by moment amplification as appropriate). The studies by Lu *et al.* and by Liapunov, and the provisions within EC3, focus on both eliminating the calculation of effective length (or using $K \leq 1$) as well as basing the design calculations on first-order forces. The AS4100 and NEHRP provisions are directed solely at the limits of applicability of first-order analyses. It can be argued that for static design, the computation of second-order elastic forces, either by approximate amplification formulae that do not require $K$ factors or by direct second-order elastic analysis, is relatively easy compared to calculation of column effective lengths. The calculation of 'appropriate' effective length factors can involve considerable complexity and design effort in certain cases, and general procedures such as system buckling analysis do not necessarily produce proper $K$ values for design[7]. The range of frames that can be designed based on $K = 1$, but using second-order elastic forces is, of course, greater than if both second-order amplification and effective length are neglected. Therefore, the authors suggest that it is most useful, in the context of static elastic analysis and design, to assess the conditions under which a design using second-order forces can be based on $K = 1$. The remaining sections of the paper address this issue.

## 3. Key concepts for assessment of when $K = 1$ is acceptable

The key concepts for assessment of when $K = 1$ is acceptable in LRFD, as well as in the design of steel frames in general, are quite simple. Use of $K = 1$ will tend to be acceptable when any of the following occur

- $KL/r$ is small, since the column strength varies little with large variations (or large error) in the effective length factor for small $KL/r$
- The columns are heavily restrained at each end, and subjected to nearly full-reversed curvature bending under sidesway of the frame, provided the gravity loads are small in any framing that leans on the lateral resisting system. Of course, if the ends of a column subjected to sidesway are prevented from rotation, and if the leaning column loads are zero, the exact solution for the effective length factor is $K = 1$
- Sufficient sidesway stiffness and strength is provided by some means in addition to the sidesway bending behaviour of the frame members
- The beam-column interaction check for the lateral resisting columns is dominated by the moment term, with the contribution from the axial strength ratio $P_u/\phi_c P_n$ being relatively small

It is important to recognize that the only error of importance in the context of a design evaluation is the error in the value obtained from the beam-column interaction equations. For any combination of the above beneficial situations, the error in the evaluation of the beam-column strength may be small even though the error in the calculated effective length factor may be very large.

## 4. Error equations associated with use of $K = 1$ in LRFD

### *4.1. Buckling model utilized for development of error equations*

Although consideration of inelastic effective length, or use of an inelastic buckling analysis, leads to substantial economy in many practical situations[18], the influence of column inelasticity on the effective length becomes small in the limit that the columns are restrained by heavy beam members, or in the unusual situation that the storey buckling is elastic. Therefore, for reasons of generality and simplicity, the error equations developed in this paper are based on elastic storey-buckling analysis and the corresponding column effective lengths. Use of an elastic storey-buckling model allows development of explicit relationships between design parameters (such as the elastic storey drift ratio and the storey-sidesway amplifier $B_2$) and the error in the beam-column interaction equations caused by use of $K = 1$. The error equations to be developed are applicable for any frames in which elastic storey-based effective length factors are acceptable for the design assessment.

LeMessurier[19] has presented a useful procedure for storey buckling analysis and determination of the associated effective lengths. Essentially all of the storey-based approaches that have been published then are a form of the equations in LeMessurier's paper[19]. This procedure has been shown to provide an excellent assessment of the effective length factors for ordinary types of steel frames in which the sidesway deflections involve predominantly shear racking of the storeys[7,19]. It is used in this paper for calculating the column strength for comparison with the strength based on $K = 1$. LeMessurier's equations for the elastic storey-based effective length may be expressed as

$$K_{C_L} = \sqrt{\frac{1}{\frac{P_u}{P_y}\left(\frac{L}{r}\right)^2} \frac{\pi^2 E}{F_y} \frac{\sum_{all} P_u + \sum_{nonleaner} C_L P_u}{\sum_{nonleaner} P_L}} = \sqrt{\frac{1}{\frac{P_u}{P_y}\left(\frac{L}{r}\right)^2} \frac{\pi^2 E}{F_y} \frac{\sum_{all} P_u}{\sum_{nonleaner} P_L} [1 + (C_L)_{avg}]} \quad (5)$$

where

$$\sum_{nonleaner} P_L = \frac{\sum_{nonleaner} HL}{\Delta_{oh}} \quad (6a)$$

is the sum of the column first-order sidesway stiffnesses associated with a unit rotation of the column chords. This term also may be written in the form

$$\sum_{nonleaner} P_L = \sum_{nonleaner} \frac{\beta EI}{L^2} \quad (6b)$$

where $\beta$ is a stiffness coefficient for each of the columns, which can be expressed as a function of the end rotational restraints provided by the girders[7,19]. The term $C_L$ in equation (5) is the $P\delta$ stiffness reduction factor, which accounts for the influence of $P\delta$ effects on the storey sidesway stability, and

$$(C_L)_{avg} = \frac{\sum\limits_{nonleaner} C_L P_u}{\sum\limits_{all} P_u} \tag{7}$$

is the weighted average of the $C_L$ values over all the columns within the storey (the $C_L$ values for leaning columns being equal to zero). For purposes of generality, one should note that equations (5)–(7) as well as equations (2) and (4) are valid only for storeys with columns of equal height. For unequal height columns, the correct versions of these equations and of the equations to follow are obtained by dividing all terms within the summations by the individual column lengths[7].

The column axial force at incipient elastic storey buckling corresponding to equation (5) is

$$P_{e(C_L)} = \frac{\sum\limits_{nonleaner} P_L}{\sum\limits_{all} P_u + \sum\limits_{nonleaner} C_L P_u} P_u = \frac{\sum\limits_{nonleaner} P_L}{\sum\limits_{all} P_u} \frac{1}{[1 + (C_L)_{avg}]} P_u \tag{8}$$

In the AISC LRFD Commentary[9], equation (8) is written in the form

$$P_{e(R_L)} = \frac{\sum\limits_{nonleaner} P_L}{\sum\limits_{all} P_u} (0.85 + 0.15\, R_L)\, P_u \tag{9}$$

by approximating $1/[1 + (C_L)_{avg}]$ by $(0.85 + 0.15\, R_L)$, where

$$R_L = \frac{\sum\limits_{leaner} P_u}{\sum\limits_{all} P_u} \tag{10}$$

For a storey that does not have any leaning columns, this simplification involves the assumption that $(C_L)_{avg}$ is equal to 0.176. This assumption is expected to be conservative for the majority of frames considered in practice, and it is adopted in this paper for estimating the errors associated with the use of $K = 1$. However, it can be violated in extreme caases, where some of the columns have

(1) Negative end restraint (i.e., negative $G$ values[7,9], in which case the column end rotations in a first-order sidesway analysis are in the opposite direction to the column chord rotations associated with the sidesway, and/or
(2) Inordinately large $L\sqrt{P/EI}$ values (for $L\sqrt{P/EI} > \pi$, or $P/(\pi^2 EI/L^2) > 1$, the $C_L$ effect can be larger than 0.216, the maximum value discussed by LeMessurier[19])

These situations can occur if a column, framed by moment-resisting connections, is quite flexible relative to the other columns and is loaded by large axial forces relative to $\pi^2 EI/L^2$. LeMessurier[20] developed an equation for the commentary of the AISC LRFD Specification which ensures that any unconservative error in the calculation of the elastic storey buckling load caused by the approximation of $1/[1 + (C_L)_{avg}]$ by $(0.85 + 0.15\, R_L)$ is negligible. This formula may be expressed in terms of the parameters $P_u$ and $P_L$ as[7]

$$\frac{P_u}{P_L} \le 1.7 \frac{\sum\limits_{all} P_u}{\sum\limits_{nonleaner} P_L} \frac{1}{(0.85 + 0.15 R_L)} \tag{11a}$$

or in terms of the column axial and shear forces as

$$\frac{P_u}{\sum\limits_{all} P_u} \le 1.7 \frac{H}{\sum\limits_{nonleaner} H} \frac{1}{(0.85 + 0.15 R_L)} \tag{11b}$$

Equation (11$b$) indicates that the error in the estimate of the elastic storey-buckling capacity, $\sum\limits_{nonleaner} P_L \; (0.85 + 0.15\, R_L)$, can become significant and unconservative when the fraction of the storey vertical load supported by a column is substantially larger than the fraction of the storey shear resisted by that member in a first-order lateral load analysis. This can happen, for example, when some of the columns are turned in weak-axis bending whereas others are oriented in strong-axis bending within the plane of the frame. Fortunately, potential violation of these equations can be determined by inspection in most practical cases.

Equations for the error associated with the use of $K = 1$ are developed below in terms of equations (5) and (8). This permits consideration of the errors in general terms, with the influence of the $P\delta$ effects addressed by selection of a value of $(C_L)_{avg}$.

### 4.2. *Relationship between errors in column and in beam-column strengths*

The nominal strength of a member as a concentrically-loaded column based on $K = 1$ may be related to the column strength based on its calculated effective length as

$$P_{n(L)} = (1 + e) P_n \tag{12a}$$

or vice versa

$$P_n = \frac{1}{(1 + e)} P_{n(L)} \tag{12b}$$

where $e$ is the error in the column axial strength associated with the use of $K = 1$, $P_{n(L)}$ is the nominal axial strength of the member based on $K = 1$, and $P_n$ is the value of the column strength based on the buckling model summarized in the previous section. It is desired to develop equations for the error in the AISC LRFD beam-column interaction equations as a function of the error in the column axial strength, $e$. The error in the LRFD beam-column interaction equations associated with the use of $K = 1$ is expressed here by $\epsilon$. If it is assumed that both $P_u/(\phi_c P_{n(L)})$ and $P_u/(\phi_c P_n)$ are greater than or equal to 0.2, equation (H1-1a) of the LRFD Specification is the governing interaction equation both for design with $K = 1$ and with the calculated effective length factor. Therefore, for a specified allowable error $\epsilon$, the effective length based interaction equation can be written as

$$\frac{P_u}{\phi_c P_n} + \frac{8}{9}\frac{M_u}{\phi_b M_n} = (1+\epsilon) \tag{13}$$

at the limit of the design resistance. Correspondingly, if the design is based on $K = 1$, and $P_u/(\phi_c P_{n(L)})$ is greater than or equal to 0.2, the engineer would use the interaction equation

$$\frac{P_u}{\phi_c P_{n(L)}} + \frac{8}{9}\frac{M_u}{\phi_b M_n} \le 1 \tag{14a}$$

An equation for $\epsilon$ independent of $\phi_c P_n$ and $\phi_b M_n$ can be obtained by dividing equation (13) by $(1+\epsilon)$, and substituting the resulting equation into the right-hand side of equation (14$a$) to obtain

$$\frac{P_u}{\phi_c P_{n(L)}} + \frac{8}{9}\frac{M_u}{\phi_b M_n} \le \frac{1}{(1+\epsilon)}\left(\frac{P_u}{\phi_c P_n} + \frac{8}{9}\frac{M_u}{\phi_b M_n}\right) \tag{14b}$$

Then, by substituting equation (12$b$) for $P_n$ into equation (14$b$) and solving this equation for $\epsilon$, the following relationship between $\epsilon$ and $e$ is obtained

$$\epsilon \le \frac{eP_u}{\phi_c P_{n(L)}} \tag{15}$$

This equation shows that if the beam-column interaction is dominated by its moment term such that the axial strength ratio is small, substantial error in the column axial strength ($e$) can be tolerated with relatively small resulting error in the value of the beam-column interaction check ($\epsilon$).

If both $P_u/(\phi_c P_{n(L)})$ and $P_u/(\phi_c P_n)$ are less than 0.2, the equivalent relationship between $e$ and $\epsilon$ is

$$\epsilon \le \frac{eP_u}{2\phi_c P_{n(L)}} \tag{16}$$

Furthermore, if $P_u/(\phi_c P_n)$ is greater than or equal to 0.2, but $P_u/(\phi_c P_{n(L)})$ (which is less than $P_u/(\phi_c P_n)$, assuming that the effective length factor is greater than one, and thus that $e$ is positive) is less than 0.2, the relationship between $e$ and $\epsilon$ is

$$\epsilon \le \left[\frac{5}{9} + e\right]\frac{P_u}{\phi_c P_{n(L)}} - \frac{1}{9} \tag{17}$$

Based on equations (14)–(17), it can be concluded that for a given error in the column strength $e$, the maximum value of the error in the beam-column interaction check, $\epsilon$, will occur when equation (14$a$) is equal to one. Furthermore, it can be concluded that $\epsilon$ depends only on $e$ and the ratio $P_u/(\phi_c P_{n(L)})$, or since $P_{n(L)}$ is a function of $P_y$ (or $F_y$) and $L/r$, the error in the beam-column interaction equations using $K = 1$ is solely a function of $e$ and the fundamental design parameters $P_u/P_y$, $L/r$ and $F_y$.

*4.3. Error in column axial strength*

The previous section developed relationships between the error in the beam-column interaction equations, $\epsilon$, and the error in the column strength based on $K = 1$, $e$. In this section, expressions for $e$ are obtained simply by considering the ratio $P_{n(L)}/P_n$, which is equal to $(1+e)$ as specified by equation (12). In general, each of the values $P_{n(L)}$ and $P_n$ may be controlled either by the AISC LRFD inelastic or elastic column strength equations. If both values are greater than or equal to 0.39 $P_y$, then they are each governed by equation (E2-2) of the AISC LRFD Specification[9] and we may write

$$(1+e) = \frac{P_{n(L)}}{P_n} = \frac{0.658^{\lambda^2_{c(L)}}}{0.658^{\frac{P_y}{P_u}\frac{\sum_{all} P_u}{\sum_{nonleaner} P_L}[1+(C_L)_{avg}]}} \tag{18}$$

where

$$\lambda^2_{c(L)} = \frac{1}{\pi^2}\left(\frac{L}{r}\right)^2\frac{F_y}{E} \tag{19}$$

If both $P_{n(L)}$ and $P_n$ are less than 0.39 $P_y$, then equation (E2-3) of LRFD controls, and the error relationship is

$$(1+e) = \frac{P_{n(L)}}{P_n} = \frac{\frac{P_y}{P_u}\frac{\sum_{all} P_u}{\sum_{nonleaner} P_L}[1+(C_L)_{avg}]}{\lambda^2_{c(L)}} \tag{20}$$

Otherwise, $P_{n(L)}$ is controlled by equation (E2-2) and $P_n$ is controlled by equation (E2-3) (assuming positive $e$), and the ratio of the two column strengths is

$$(1+e) = \frac{P_{n(L)}}{P_n} = \frac{0.658^{\lambda^2_{c(L)}}}{0.877}\frac{P_y}{P_u}\frac{\sum_{all} P_u}{\sum_{nonleaner} P_L}[1+(C_L)_{avg}] \tag{21}$$

By inspection of the above equations, it can be seen that the error $e$ depends on $P_u/P_y$, $L/r$, $F_y$, $\sum_{all} P_u / \sum_{nonleaner} P_L$, and $(C_L)_{avg}$. Furthermore, the term $\sum_{all} P_u / \sum_{nonleaner} P_L$ is directly related to $B_2$ (equation (4)), and may be written in terms of this parameter as

$$\frac{\sum_{all} P_u}{\sum_{nonleaner} P_L} = \frac{B_2 - 1}{B_2} \tag{22}$$

It is felt that the values of elastic sidesway amplifier $B_2$ are in general more familiar than values of $\sum_{all} P_u / \sum_{nonleaner} P_L$. Therefore, equation (22) is used here to express the error $e$ in terms of $B_2$.

*4.4. Overview of procedure for calculating $\epsilon$*

The equations presented above show that the error in the beam-column interaction equations $\epsilon$ (associated with equation (14$a$) equal to one) can be expressed generally as a function of $B_2$, $(C_L)_{avg}$, $F_y$, $L/r$, and $P_u/P_y$. The basic procedure for calculating this error is: (1) Determine $e$ from equations (18)–(22); and (2) Compute $\epsilon$ from equations (15)–(17).

The results for $B_2 = 1.11$, $(C_L)_{avg} = 0.176$, and $F_y = 250$ MPa (36 ksi) are plotted for various $L/r$ and $P_u/P_y$ values in *Figure 1*. The $B_2$ value of 1.11 is chosen because this is the limit proposed in Eurocode 3[5] for consideration of a frame as 'nonsway', and it is approximately the limit suggested in AS4100[6] for which frame stability effects may be neglected in plastic design. A subsequent section of the paper shows that the errors generally increase with increases in $B_2$. The value $(C_L)_{avg} = 0.176$ is an estimate of the maximum potential reduction in the storey elastic buckling capacity due to $P\delta$ effects, as previously discussed. The value of $F_y = 250$ MPa is a practical lower bound to the yield strengths used in current building construction (it is shown later in the paper that the error is generally reduced for larger values of $F_y$).

The specific behaviour underlying *Figure 1* is described in the next section. This is followed by a broader discussion of the influence of the different variables on the error $\epsilon$.

## 5. Key error attributes and influence of $L/r$ and $P_u/P_y$ on $\epsilon$

Earlier in the paper, general concepts for when the use of $K = 1$ is appropriate have been outlined. However, a major goal of this paper is to determine specific upper-bounds on the errors associated with assuming $K = 1$. The following are key characteristics associated with the plot in *Figure 1*. These characteristics provide detailed insight into when it is appropriate to use $K = 1$, and they form the basis for specific $K = 1$ guidelines outlined later.

- Based on *Figure 1*, the error $\epsilon$ may be determined for specified values of $L/r$ and $P_u/P_y$. However, there are an infinite number of possible frames associated with any one of the points in the plot. This is a result of the fact that equations (5)–(8) do not require specification of any values for the column rotational end restraint or for the leaning effects on the storey being considered.
- Larger errors occur for the curves with the smallest $L/r$ values. This is because, if $B_2$, $(C_L)_{avg}$, $F_y$ and $P_u/P_y$ are held constant, then as a column becomes stiffer by having its $L/r$ reduced, it will participate to a greater extent in stabilizing the storey. Thus its actual effective length will increase.
- For larger $L/r$ and $P_u/P_y$ values, the column tends more and more towards being 'braced' by the other storey framing. At the points where each of the curves intersect the horizontal axis in *Figure 1*, the error is equal to zero because the value of $K_{C_L}$ (equation (5)) is equal to one. For example, the error plots in the figure indicate that a column with $L/r = 50$ would have a value of $K_{C_L} = 1$, and therefore zero error, if $P_u/P_y$ is equal to approximately 0.58.
- The curves for each of the $L/r$ values exhibit the largest error when the axial load to yield load ratio $P_u/P_y$ is near zero. However, the error reduces rapidly with increases in $P_u/P_y$ for values of this parameter less than approximately 0.11. Most of the curves reach a local minimum error approximately at $P_u/P_y = 0.11$, and then show a rapid increase in the error over a short range of $P_u/P_y$ as the axial load to yield load ratio is increased further. The error for these curves peaks again at $P_u/P_y$ between 0.11 and 0.17, the location of this local maximum depending on which $L/r$ curve is considered. For $P_u/P_y$ larger than 0.17, all the curves show a decrease in the error with increasing values of the axial load to yield load ratio.
- The reasons for the above variation in the error with $P_u/P_y$ are as follows. The tendency for the error to reduce with increasing values of $P_u/P_y$ has already been discussed, and is due to the fact that for larger values of $P_u/P_y$, the column participates less and less in stabilizing the storey and actually may start to 'lean' on the other storey framing. The dramatic dip in the error curves for small $P_u/P_y$, shown in *Figure 1*, is due to changes in the controlling beam-column interaction equations. For $P_u/P_y < 0.11$, AISC LRFD equation (H1-1b) governs the interaction checks based on both $P_n$ and $P_{n(L)}$ for all the curves shown in the figure (i.e., the relationship between $e$ and $\epsilon$ is given by equation (16)). However, for values of $P_u/P_y$ between 0.11 and 0.17, LRFD equation (H1-1b) controls in many cases for the interaction check based on $P_{n(L)}$, whereas equation (H1-1a) (equation (13)) controls for the check based on $P_n$. As a result, equation (17) is utilized in calculating the error for these points. For $P_u/P_y \geq 0.17$, equation (H1-1a) always controls for both beam-column interactionn checks (i.e., equations (13) and (14*a*) are both applicable), and $e$ and $\epsilon$ are related by equation (15). For the curves that show a local maximum at $P_u/P_y = 0.13$ to 0.17 (all the curves except for $L/r = 80$ and 90), the local maximum corresponds to the lowest value of $P_u/P_y$ where equation (H1-1a) controls for both interaction checks. That is, the local maximum error at $P_u/P_y = 0.13$ to 0.17 is associated with the 'knee' in the beam-column interaction curve based on $K = 1$[9].
- When $P_u/P_y$ is larger than approximately 0.05, the AISC-LRFD inelastic column strength equation, (E2-2), governs the calculation of both $P_n$ and $P_{n(L)}$ for all the curves shown in *Figure 1*. Therefore, the elastic column strength equation (LRFD equation (E2-3)) and the associated error equations (equations (20) and (21)) have no influence for the majority of points in the figure. Of course, this is expected in general since it is rare that the strength of a frame would be controlled by elastic storey buckling. Also, the types of frames corresponding to $P_u/P_y < 0.05$ in *Figure 1* tend to have extremely large leaning column effects, with only a few lightly-loaded lateral-resisting columns acting as springs that are propping up the leaning columns of the storey. As a result, the errors shown in *Figure 1* for $P_u/P_y$ less than about 0.05 are not considered to be of any practical sig-

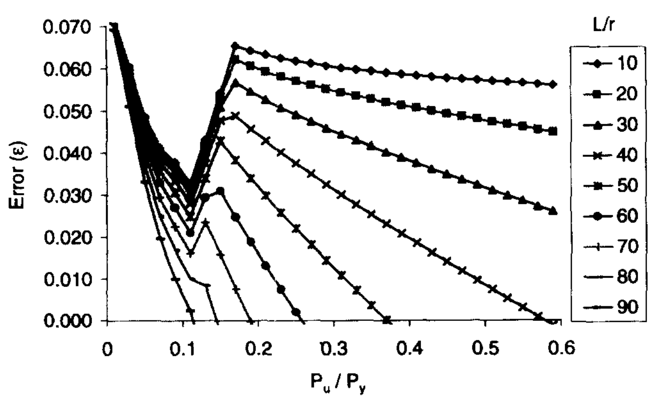


*Figure 1* Error in beam-column interaction equations ($e$) for $B_2 = 1.11$, $(C_L)_{avg} = 0.176$ and $F_y = 250$ MPa

nificance. The use of $K = 1$ for situations in which the error is larger than that predicted by a maximum error equation developed subsequently can be disallowed (for any $B_2$) by limiting this simplification to frames in which the vertical load capacity is controlled by inelastic storey buckling. This requirement can be checked by inspection in most cases, or in general by using the simple approximate equation

$$\left[ S_L = \frac{\sum\limits_{nonleaner} P_y}{\sum\limits_{nonleaner} P_L} N \right] \leq 2.25 \tag{23}$$

where

$$N = \frac{\sum\limits_{all} P_u}{\sum\limits_{nonleaner} P_u} = \frac{1}{1 - R_L} \tag{24}$$

The parameter $S_L$ is an estimate of the sum of the yield loads for all the columns of the storey to the elastic storey buckling capacity, and is used by LeMessurier[18] in a similar way to limit the application of simplified equations he has developed for calculating column effective length.

Two observations can be made based on *Figure 1* that have important implications with respect to practical restrictions for the use of $K = 1$ in design: firstly, since for $P_u/P_y \geq 0.17$, the errors decrease with increasing $P_u/P_y$ for all values of $L/r$, there appears to be no need to restrict the maximum value of $P_u/P_y$ as suggested in Reference 21 to limit the error associated with using $K = 1$. Secondly, the error plot demonstrates that there is no need to restrict the maximum $L/r$ of the columns to control the error associated with using $K = 1$. By placing a limit on $B_2$, then: (1) the slenderness of at least some of the columns within the storey must be sufficiently small; (2) the total gravity loads on the storey must be sufficiently small, or (3) an alternative device that gives adequate storey lateral stiffness (such as sidesway bracing) must be provided.

## 6. Upper-bound estimate of the error for $B_2 \leq 1.11$

Based on *Figure 1*, a 'practical worst case' error can be estimated for any frame with $B_2 \leq 1.11$, $F_y \geq 250$ MPa, and $(C_L)_{avg} \leq 0.176$ (the error reduces with larger $F_y$ and smaller $(C_L)_{avg}$, as discussed below). If $L/r = 10$ is assumed as a lower-bound for the column slenderness, the maximum error for $P_u/P_y \geq 0.05$ is 0.065, or 6.5% (at $P_u/P_y = 0.17$). The value of $K_{C_L}$ corresponding to this point is 7.38. This type of situation can only be achieved with very large leaning column effects, and/or very low rotational end restraint (in which case, it is practically impossible to obtain $(C_L)_{avg} = 0.176$). Therefore, the above estimate of the maximum $\epsilon$ is conservative. Furthermore, $L/r$ values this low generally require the use of a beam-type section. It can be argued that $L/r \cong 20$ is probably a better practical lower bound for steel frame design. If $L/r = 20$ is assumed as a lower-bound value for the column slenderness, the maximum error in *Figure 1* for $P_u/P_y \geq 0.05$ is reduced to 0.062 (at $P_u/P_y = 0.17$), with a corresponding value of $K_{C_L} = 3.69$. Therefore, one may conclude that for $B_2 \leq 1.11$ and $F_y \geq 250$ N/mm$^2$, 6% is a reasonable upper-bound estimate for the error in the AISC LRFD beam-column interaction equations associated with the use of $K = 1$.

## 7. Miscellaneous effects on the maximum error for $B_2 \leq 1.11$

The previous sections have studied in detail the error attributes associated with neglecting the effective length for a wide range of $L/r$ and $P_u/P_y$ values, and for constant parameters $B_2 = 1.11$, $(C_L)_{avg} = 0.176$, and $F_y = 250$ MPa. The discussion concludes by estimating that an upper bound for this error is 6%. This section addresses a number of miscellaneous factors that can cause the actual error to be less than this estimate. It focuses first on reductions in the error due to variations in $(C_L)_{avg}$ and $F_y$ from the values associated with the maximum error estimate. The effects of these parameters may be evaluated from equations (15)–(22). Subsequently, reductions in the error caused by several factors that do not appear in the above error equations are discussed. The reader is referred to Reference 7 for a more detailed exposition. Of the easily quantified design parameters, $L/r$, $P_u/P_y$ and $B_2$ have the most significant effect on the error. The influence of $L/r$ and $P_u/P_y$ has already been discussed. The influence of $B_2$ is addressed in a separate and subsequent section.

### 7.1. Pδ *stiffness reduction*, $(C_L)_{avg}$

The value $(C_L)_{avg} = 0.176$ is practically impossible for the types of frames that produce errors close to the 6% upper bound. A value of $(C_L)_{avg}$ equal to 0.056 is representative of designs based on the maximum error parameters (i.e. $L/r = 10$ or 20 and $P_u/P_y = 0.17$)[7]. Specific example frames in which $(C_L)_{avg} = 0.056$ are shown later in this paper. For $B_2 \leq 1.11$, $F_y \geq 250$ MPa, and $(C_L)_{avg} = 0.056$, the maximum errors based on the elastic storey-buckling model are reduced to 5.8 and 5.5% for $L/r = 10$ and 20, respectively.

### 7.2. *Yield stress*, $F_y$

If the case $B_2 = 1.11$, $(C_L)_{avg} = 0.176$, and $F_y = 350$ MPa (50 ksi) is considered, the maximum error is still 6.5% for $L/r = 10$. It is reduced slightly to 6.1% for $L/r = 20$. These errors still occur at $P_u/P_y = 0.17$. For larger $L/r$ and $P_u/P_y$ values, the reduction in the error by increasing the yield stress from 250 to 350 MPa is significantly greater than in *Figure 1* (see Reference 7). Nevertheless, $F_y$ has little effect on the maximum error in using $K = 1$.

### 7.3. *Beam-columns controlled by out-of-plane strength*

For all the cases that have been considered thus far, it has been assumed that the design is controlled by in-plane stability. In many cases, a building frame may have its columns subjected to sidesway and oriented in strong-axis bending in-plane, but effectively braced at each of the floor levels in the out-of-plane direction. In this situation, $P_n$ is often controlled by the out-of-plane strength. This is the case if $K_yL_y$ is greater than $K_xL_x/(r_x/r_y)$. A reasonable lower-bound for the $r_x/r_y$ of rolled wide-flange shapes is 1.6. Therefore, if we assume that an effective length $K_y = 1$ is used for the out-of-plane check, and that $L_x = L_y = L$, the weak-axis column strength would control the design of wide-flange columns unless $K_x$ is greater than 1.6.

If the design is based on $K = 1.6$ rather than $K = 1$, the maximum error for $L/r = 10$ is reduced only to 6.4% at $P_u/P_y = 0.17$ (a change of only 0.1% from *Figure 1*); the corresponding effective length factor is 4.61. For $L/r = 20$, the maximum error is 5.5% (reduced from 6.2% in *Figure 1*). The error equations for this problem are identical to those presented with the exception that equation (19) is multiplied by $K^2 = 1.6^2 = 2.56$. Essentially, for the critical low $L/r$ cases, the leaning column effects are so high, and/or the column end rotational restraint is so small when attaining the maximum error, that designing the column with $K = 1.6$ versus $K = 1$ has only a minor effect on this error.

It is interesting to consider the errors obtained if all the beneficial effects discussed thus far are combined. If the design parameters are $B_2 \leq 1.11$, $F_y \leq 350$ MPa, $(C_L)_{avg} \leq 0.056$ and $K = 1.6$, then the maximum errors become 5.5% and 4.3% for $L/r = 10$ and 20, respectively.

### 7.4. *Redundancy and inelastic reserve strength*

In building frame designs, there is usually some redundancy and an associated amount of inelastic reserve strength within the structural system. This attribute of the behaviour was relied upon in the prior research outlined in Reference 1. Inelastic force redistribution can increase the strength of framing systems far above that estimated by elastic design procedures. Furthermore, it is present in most frames that have a reasonable degree of redundancy. However, it cannot be counted upon generally unless a second-order inelastic analysis check (such as is discussed in Reference 21) is utilized, and thus, it is not incorporated into the error estimates presented here.

### 7.5 *Column inelastic stiffness reduction and reversed curvature bending*

As previously discussed, the effective length is reduced in many practical situations due to distributed yielding and the resulting inelastic stiffness reduction within the columns. This reduction can be quite large in cases where the elastic rotational restraint from the girders and beam-to-column connections is relatively small. However, accurate determination of the inelastic buckling capacity requires, in general, an iterative analysis, and development of explicit relations between $B_2$ and $\epsilon$ based on an inelastic buckling model is not possible.

Also, as noted previously, it is expected that when the columns of a frame are subjected to nearly full-reversed curvature bending under sidesway (i.e., approximately equal and opposite-sign end moments due to sidesway deflections), the possibility of basing the design on $K = 1$ is increased. This aspect of the behaviour also is not reflected directly in the error relationships that have been developed.

The influence of column inelasticity and reversed-curvature bending on the error can be considered only by studying specific benchmark frames. Studies of this nature have been presented[7] for practical extreme values of end rotational restraint associated with $G \cong 0.25$, and assuming minimum values of $F_y = 250$ MPa, $L/r = 20$, and $B_2 = 1.11$ (the factor $G$ is equal to the $\Sigma(I_c/L_c)/\Sigma(I_g/L_g')$ at the end of a column, where $L_g' = L_g\,(2 - M_F/M_N)$ is the equivalent girder length for calculation of effective length factors from the AISC LRFD sidesway-uninhibited alignment chart, and $M_F$ and $M_N$ are the moments due to sidesway, located at the near and far ends of the girder relative to the column[9]. These studies show that the maximum error in the beam-column interaction equations compared to the strengths based on a precise inelastic effective length calculation is 5.1% if the opposite end of the column is pinned, and 4.0% if the opposite end of the column is restrained with $G \cong 1$. This latter case corresponds to an arbitrarily selected degree of reversed-curvature bending in which the inflection point under sidesway deflections is just within the middle one-seventh of the column length. Both of the above error bounds include the effects of reduced $(C_L)_{avg}$ values encountered in the maximum error cases. Furthermore, for larger $L/r$, larger $F_y$, and larger minimum $G$ values, the maximum errors can be substantially smaller. Also, if the frame has a reasonable degree of redundancy, and/or if the column axial strength is controlled by out-of-plane buckling, any overprediction of the elastic in-plane design strength by using $K = 1$ is of reduced significance.

## 8. Influence of the sidesway amplification factor $B_2$

As noted at the beginning of the previous section, $B_2$ is one of the most important parameters that influence the error in the beam-column interaction equations associated with the use of $K = 1$. *Figure 2* is the same plot as in *Figure 1*, but for $B_2 = 1.17$ instead of 1.11. The value of 1.17 is selected since this is the minimum $B_2$ limit in LRFD based on the recommendations by Lu *et al.*[10], assuming that $\sum_{all} P_u / \sum_{all} P_y$ is greater than or equal to one, as discussed previously.

Comparing *Figures 1* and *2*, it can be seen that the error curves for different $L/r$ values have the same shape in each of the plots, but the errors based on $B_2 = 1.17$ are somewhat larger. The maximum error for $B_2 = 1.17$ is 10.4% (again for $L/r = 10$ and $P_u/P_y = 0.17$). This error might be judged to be excessive.

Similar plots have been generated by the authors for other values of $B_2$. An empirical relationship between $\epsilon_{max}$ and $B_2$, which fits the maximum error data reasonably well and provides a simple means of relating the value of $B_2$ to the error in using $K = 1$ for column design, is

$$\epsilon_{max} = 0.5B_2(B_2 - 1) \qquad (25)$$

This equation slightly underestimates the upper-bound error for $B_2 \leq 1.25$ (the largest underestimate is 9.9% versus 10.4% at $B_2 = 1.17$). It increasingly overestimates the error for increasing values of $B_2 > 1.25$ (at $B_2 = 1.4$, equation (25) gives $\epsilon_{max} = 0.28$ whereas the error equations

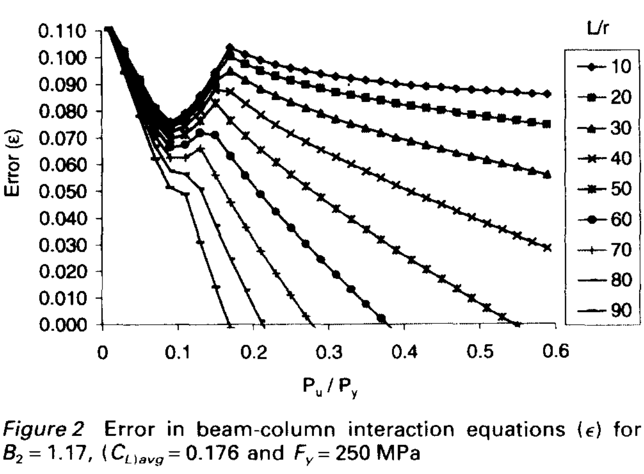


*Figure 2* Error in beam-column interaction equations ($\epsilon$) for $B_2 = 1.17$, $(C_L)_{avg} = 0.176$ and $F_y = 250$ MPa

predict a maximum error of 0.26; at $B_2 = 1.6$, this gives $\epsilon_{max} = 0.48$ whereas the error equations predict a maximum error of only 37%). Plots of these errors are given in Reference 7. It is important to understand that equation (25) is based on the following limits: $(C_L)_{avg} \leq 0.176$, $S_L \leq 2.25$, $L/r \geq 10$, and $F_y \geq 250$ MPa. It has been explained that it is practically impossible for $(C_L)_{avg}$ to exceed the above limit for the combinations of parameters that produce the maximum errors. The limit on $S_L$ prevents the application of $K = 1$ to any types of frames for which the error is larger than given by equation (25). As shown in *Figures 1* and *2*, the errors can be larger than the local maximum at $P_u/P_y = 0.13$ to 0.17 for extremely small values of $P_u/P_y$. For $B_2 > 1.4$, the dip in the error plots for $P_u/P_y < 0.13$ to 0.17 disappears such that $\epsilon$ actually decreases monotonically with increasing $P_u/P_y$ for all values of $L/r$. The above limits on $L/r$ and $F_y$ are simply minimum values that would be expected in practice.

Equation (25) is quite useful, since the allowable value of the beam-column interaction equations can be reduced based on $\epsilon_{max}$ to ensure that designs with $K = 1$ will never violate the interaction equations based on the 'correct' value of $K$. This concept is discussed next.

## 9. Limiting the interaction equation values based on $B_2$

In the development of the error plots in *Figures 1* and *2*, it is assumed that the value of the beam-column interaction equations based on $K = 1$ (e.g., equation (14*a*)) are equal to one. This is true only if the design is controlled by strength. If the design is controlled by drift or some other serviceability criterion, the interaction equation may have a value substantially less than one. The error $\epsilon$ reduces rapidly as the value of the interaction equation falls below one. For example, if the beam-column interaction check evaluates to less than approximately 0.94 and $B_2$ is less than 1.11, it can be stated that the 'practical worst case' unconservative error is zero. In other words, if the engineer wishes to ensure that his or her design is conservative, the design can still be based on $K = 1$ if $B_2$ is held less than 1.11 and the beam-column interaction checks are limited to 0.94. Alternatively, if $B_2$ is less than 1.17, the design can be based conservatively on $K = 1$ if the interaction values are limited to 0.90. In fact, a general limit on the interaction equation values can be set as $1/(1 + \epsilon_{max})$, where $\epsilon_{max}$ is given as a function of $B_2$ by equation (25).

A question arises when considering how to apply this approach in the context of beam-column design where the out-of-plane column strength may control for $P_n$. In this situation, the most appropriate procedure is to make two checks: one for the lateral-torsional strength of the member with $P_n$ based on the out-of-plane column strength and a maximum value of the interaction equation equal to 1.0, and one for in-plane strength based on $K = 1$ with a maximum allowable value less than one. At the expense of some extra conservatism in checking the lateral-torsional strength, a single interaction equation value may be computed based on the controlling $P_n$ and limited to $1/(1 + \epsilon_{max})$.

## 10. Summary of design recommendations

Based on the studies presented, it is recommended that $K = 1$ can be used in AISC LRFD with negligible error whenever $B_2$, calculated using equation (4), is less than or equal to 1.11. The upper-bound on the resulting error is approximately 6%. The actual maximum error is on the order of 5% for pinned base conditions and 4% for cases involving nearly full-reversed curvature bending, as long as $G \geq 0.25$.

If the engineer wishes to ensure that a design based on $K = 1$ is conservative, the beam-column interaction values may be limited to 0.94 as long as $B_2 \leq 1.11$. Use of $K = 1$ can be extended to frames with any characteristic $B_2$ values by limiting the beam-column interaction equations to $1/(1 + \epsilon_{max})$, where $\epsilon_{max}$ is given by equation (25).

These recommendations are based on estimates of the maximum possible error in the beam-column interaction equations associated with $B_2 = 1.11$ (or equal to the specified $B_2$ value), $L/r \geq 10$, and $F_y \geq 250$ MPa. A $(C_L)_{avg}$ of 0.176 is assumed in the maximum error calculations. It has been explained that for the combinations of parameters required to produce the maximum errors, it is practically impossible for the $P\delta$ stiffness reduction factor to exceed this value.

The above recommendations are subject to the following restrictions:

(1) Second-order elastic forces must be used in the design evaluation. These forces may be computed either by direct analysis or by appropriate amplification of first-order elastic forces. Geometric imperfections or notional load effects do not need to be considered in the force calculations. It should be noted that this restriction is very important. If the second-order amplification of the design forces and the column effective length are both neglected for the maximum error cases, the beam-column interaction equations can be up to 16% in error for $B_2 = 1.11$.

(2) The vertical load capacity of the structure must be governed by inelastic storey buckling. Equation (23) can and should be used for unusual frames where this restriction cannot be verified by inspection.

(3) The calculation of the column strengths based on elastic storey buckling, or equivalently based on effective length factors associated with elastic storey buckling, must be valid.

No restrictions on the maximum values of $L/r$ and/or $P_u/P_y$ are required.

For building frames, the second of the above restrictions is likely to always be an academic one. Also, restrictions associated with the use of elastic storey buckling equations (item (3) above) are academic for most problems. Nevertheless, they are real. The limitations of storey buckling models are often not well understood, and therefore it is appropriate to summarize some of these limits. First, if a storey has any columns that are relatively slender and heavily loaded compared to the other columns, equation (11) should be checked to ensure that the storey buckling strength is not substantially reduced by the weak column(s) (which may in fact buckle in a nonsway mode). Also, for the correct application of storey buckling equations (such as equations (5)–(8)) the axial forces in the girders must be negligible; otherwise, the girder stiffnesses must be reduced to account for these effects[7]. For slender frames, it is implicitly assumed that any reduction in the system buckling strength due to frame 'cantilever bending' actions is insignificant[7]. It is assumed in general that the buckling behaviour is a storey-by-storey phenomenon, with limited interstorey buckling interactions. For design of frames having partially-restrained connections, the connection non-linearity must be accounted for in the determination of $B_2$ by an appropriate tangent stiffness approximation[7]. Finally, engineers often anticipate that designs using elastic effective lengths are at worst somewhat conservative relative to those based on inelastic $K$ values. This is not true in general. Due to greater inelastic stiffness reduction in columns with larger $P_u/P_y$ values, the storey-based inelastic effective lengths in columns that have smaller $P_u/P_y$ can increase relative to their elastic values[7]. In certain cases, the resulting unconservative error in the beam-column interaction equations (relative to the strength based on a precise inelastic $KL$) can be significant. For cases in which the lateral resisting columns have similar $P_u/P_y$, and/or the $P_u/P_y$ values on all the lateral resisting columns are significantly less than one at the limit of the structure's resistance, and/or the structure has substantial redundancy, these unconservative errors are likely to be negligible[22]. However, for fames that: firstly, contain columns with substantially different levels of $P_u/P_y$; and secondly, have little redundancy, the column strengths may need to be based on an inelastic buckling analysis (or the corresponding inelastic effective lengths) for adequate assessment of the inelastic stability effects.

The above recommendations are particularly useful for judging when the columns in moment frames with 'partial' or 'flexible' bracing provided by diagonal members, walls, or cladding may be designed using $K = 1$. If the lateral stiffness provided by the bracing elements is denoted by the symbol $k_{bracing}$, where this stiffness is defined as the bracing force generated by a unit storey lateral drift, the effect in reducing $B_2$ may be included by adding the value $k_{bracing}L$ to the $\Sigma P_{L\,nonleaner}$ term in equation (4). Alternatively, if the columns within the storey are of unequal height, all the terms within the summations should be divided by the respective column lengths, and the lateral bracing stiffness $k_{bracing}$ should be added directly within this term of equation (4). Of course, the strength of the bracing elements also must be checked in this type of a design.

## 11. Case study frames

It is useful to consider several 'worst-case' frames to assess the accuracy of the error relationships, and to better understand the implications of using $K = 1$ relative to the rigorous theoretical inelastic frame behaviour. *Figure 3* shows two such frames, one in strong-axis bending and the other in weak-axis bending in the plane of the page. These frames have a $B_2$ of 1.25 at $P/P_y = 0.17$ in the right-hand lateral resisting column (arbitrarily selected as larger than the $B_2$ values considered in *Figures 1* and *2*). The combination of zero redundancy, pinned-base conditions, heavy leaned column effects ($N = 6.28$ from equation (24)), substantial rotational end restraint ($G = 0.243$), low slenderness of the lateral resisting column ($L/r = 20.15$), and small yield strength ($F_y = 250$ MPa) lead to an error prediction of $\epsilon = 15.6\%$ from equation (25) for the right-hand column of these frames. The W360 × 314 column shown in the figure (W14 × 211 in imperial units) is the heaviest Group 3 W shape available[9], and the storey height of 3.35 m (11 ft) for the strong-axis bending case is relatively small compared to most practical situations. Nevertheless, the $L_c/r_c$ of this column is still not less than 20. For the weak-axis case, the $L_c$ corresponding to $L_c/r_c = 20.15$ is quite small. This demonstrates that $L/r = 20$ is a reasonable lower-bound for most steel frames, as previously stated. The W840 × 527 (W33 × 354) girder with $L_b = 3.35$ m for the strong-axis case is obviously rather stocky in proportion to the column compared to most practical situations; however, this is required to restrict $B_2$ to 1.25 given the other parameters for this frame.

*11.1. Strong-axis bending case study*

*Figure 4* shows four distinct beam-column interaction curves for the example strong-axis frame, expressed in terms of the normalized axial force in the right-hand column ($P/P_y$) versus the normalized maximum primary bending moment in this member ($HL/M_p$). The bold curve in the figure corresponds to the LRFD column strength based on a precise inelastic buckling analysis outlined by LeMessurier[18]. This is referred to as the LRFD($K_{inel}$) curve in the discussions that follow. The inelastic effective length factor $K_{inel}$ associated with LeMessurier's solution is an alternative to equation (5). By including the effects of column inelastic stiffness reduction in the buckling analysis, LeMessurier's approach[18] tends to produce a slightly more liberal interaction equation than that based on equation (5) for the examples studied here.

The second curve listed in the legend is the LRFD interaction curve associated with $K = 1$. For this example, the LRFD($K = 1$) curve is practically identical to the beam-column strength obtained from a second-order elastic-plastic hinge analysis in which the LRFD beam-column strength for $L = 0$ is used as the full-plastification strength of the

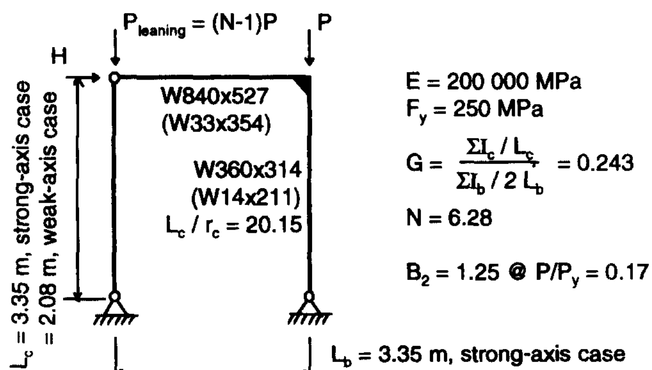


*Figure 3* Example 'maximum error' frames in strong- and weak-axis bending

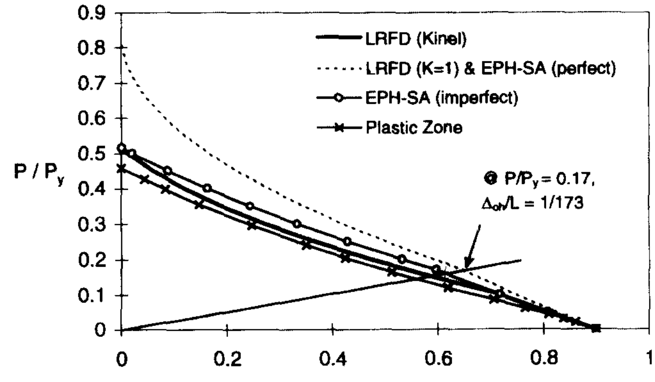


*Figure 4* Beam-column interaction curves for example frame in strong-axis bending

cross-section (the LRFD curve for $L = 0$ gives a reasonably accurate fit to the exact cross-sesction plastic strength for strong-axis bending of wide-flange sections[21]). The axial yield strength and the plastic moment capacity are set at $0.85P_y$ and $0.9M_p$ for the elastic-plastic hinge analysis, where the values 0.85 and 0.9 correspond to the AISC resistance factors for axial and bending strength, respectively.

The third curve listed in the legend is obtained by a second-order elastic-plastic hinge analysis, but with the frame assumed to be initially out-of-plumb by $L/500$. This value for the out-of-plumbness corresponds to the maximum erection tolerance on a shipping piece specified in the AISC Code of Standard Practice[9]. A comparison of the two elastic-plastic hinge curves gives an indication of the effect of geometric imperfections on the strength of the frame. Furthermore, the EPH-SA(imperfect) curve may be compared to plastic zone analysis results to ascertain the effects of distributed yielding in the lateral resisting column. As noted previously, the AISC interaction curves account for these effects solely within the calculation of the column strength $\phi_c P_n$.

The fourth curve specified in the legend is obtained by a rigorous plastic zone analysis[23]. A nominal yield strength of $F_y = 250$ MPa is used for the analysis, and the resulting beam column strengths are factored a posteriori by 0.9 along both axes. Use of this 'uniform' resistance factor might be justified with this type of analysis based on the precision with which the nominal effects on the design strength can be modelled. A residual stress pattern presented by Galambos and Ketter[24], with compressive values at the flange tips of $0.3F_y$, is assumed for the plastic zone analysis. The stress–strain behaviour is assumed to be elastic-perfectly plastic (no strain hardening). Also, an out-of-plumbness of $L/500$ and an out-of-straightness of $L/1000$ are assumed, with the lateral-resisting column being bowed to the right to create a detrimental out-of-straightness effect. Plastic zone analyses with these geometric imperfections and residual stresses have been shown to reproduce closely the AISC-LRFD column strengths based on inelastic effective length for light-to-medium weight column W sections and a wide range of end restraint conditions[7].

A radial line from the origin, which intersects the $K = 1$ strength curve at $P/P_y = 0.17$ (this point corresponding to $B_2 = 1.25$), is shown in *Figure 4*. The error in the LRFD ($K = 1$) curve, measured along this line, is 12% relative to the LRFD($K_{inel}$) curve, 6% relative to the EPH-SA(imperfect) curve, and 16% relative to the plastic zone strength.

The following key observations can be drawn from *Figure 4*:

- The LRFD($K_{inel}$) curve is slightly unconservative, but fits closely with the plastic zone strength.
- The lengths based on $K = 1$, which are essentially the same as the strength based on a second-order elastic-plastic hinge analysis without initial imperfections, significantly overpredict the plastic zone strengths. For the point on the $K = 1$ interaction curve corresponding to $B_2 = 1.25$, the error measured along the radial line compares reasonably well with the predicted $\epsilon_{max}$ value from equation (25). Of course, it should be noted that $\epsilon$ is defined as the fraction by which the 'correct' LRFD interaction check exceeds the value 1.0. This does not match exactly with the definition of the error based on the ratio of the distances to the curves along a radial line from the origin of the plots, but the differences are minor.
- The strengths predicted by second-order elastic plastic hinge analysis, including imperfections, do not adequately capture the rigorous plastic zone strengths either (although the error is much reduced from that of the EPH-SA(perfect) curve). This is primarily due to the fact that the elastic-plastic hinge analysis does not account for beam-column distributed plasticity effects. The LRFD($K_{inel}$) curve accounts for this behaviour by basing $\phi_c P_n$ on an iterative inelastic buckling analysis, and by specifying a simple function for the interaction between $\phi_c P_n$ and the column's bending capacity.
- The error in the beam-column interaction curves relative to the plastic zone strengths increases as the column axial loads are proportionally increased such that $P/P_y$ is greater than 0.17. However, $B_2$ also increases as the column axial loads are increased, such that the estimated error $\epsilon$ is also greater. For any of the points corresponding to $P/P_y > 0.17$, the estimated error in the LRFD ($K = 1$) curve based on equation (25) is generally larger than the actual error measured along a radial line from the origin to the point under consideration.
- The first-order drift ratio at the factored load level corresponding to $P/P_y = 0.17$ is relatively large (1/173) for the example frame. Obviously, this drift is probably too high for the frame to be serviceable under nominal lateral loads. Nevertheless, the frame can be checked using $K = 1$ as long as the interaction equation value is limited to $1/(1 + \epsilon_{max}) = 0.86$. For larger axial and smaller lateral load, the drift is, of course, smaller. However, $B_2$ is larger, and thus the corresponding limit on the interaction equation value for use of $K = 1$ must be reduced to restrict the error to acceptable values. This indicates that limiting the first-order drift ratio alone is not sufficient to ensure that the error associated with the use of $K = 1$ is small.

### *11.2. Weak-axis bending case study*

The above example considers only strong-axis bending. It is important also to assess the behaviour for weak-axis cases, since the inelastic flexural stiffness tends to reduce more dramatically as the flange tips become yielded in weak-axis bending. *Figure 5* shows results comparable to *Figure 4* but for the weak-axis bending frame of *Figure 3*. The same column cross-section (W360 × 314) is used for this problem. Also, a W840 × 527 girder is used in strong-

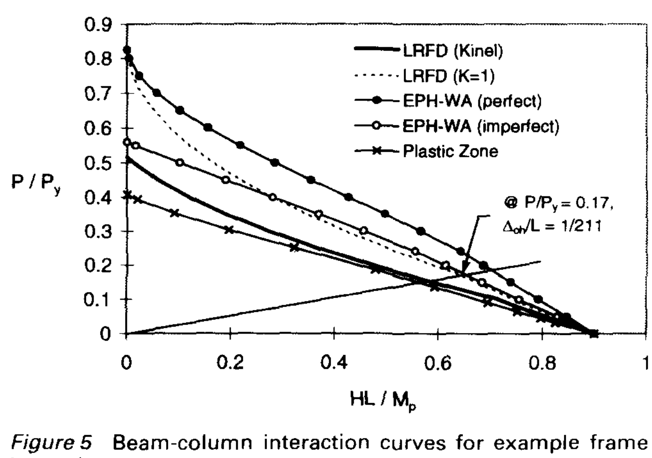


*Figure 5* Beam-column interaction curves for example frame in weak-axis bending

axis bending, but the girder length is increased to obtain the specified $G$ value. The weak-axis cross-section full-plastification strength specified in EC3[5] is used for calculation of the second-order elastic-plastic hinge curves.

The behaviour in *Figure 5* is similar to that shown in *Figure 4* with some notable exceptions. The LRFD($K_{inel}$) and ($K = 1$) curves in *Figure 5* are identical to the corresponding curves for the strong-axis bending case. The plastic-zone strength curve is similar to the one for strong-axis bending, but due to the sharper reduction in the section stiffness with yielding at the flange tips, the weak-axis strengths are somewhat smaller for $P/P_y$ larger than about 0.25. This is in spite of the fact that the cross-section full-plastification strength for weak-axis bending is quite convex. The convexity of the weak-axis cross-section strength is apparent in the results of the elastic-plastic hinge analyses. Both of the elastic plastic hinge curves (with and without an initial out-of-plumbness of $L/500$) are significantly unconservative compared to the rigorous plastic zone solution.

For the point on the LRFD($K = 1$) curve corresponding to $B_2 = 1.25$ and $P/P_y = 0.17$, the error relative to the plastic zone solution is approximately 14%, which is close to the predicted $\epsilon_{max}$ of 15.6% from equation (25) and to the 16% error for the corresponding strong-axis bending case. For larger $B_2$ (i.e., larger $P/P_y$ values), the error in the LRFD($K = 1$) curve for this problem is larger than that for the strong-axis case, but it is still estimated reasonably well by equation (25). This larger error is associated in part with the use of a single column strength curve in LRFD. The LRFD($K_{inel}$) curve fits the plastic zone strength well in this problem for $P/P_y \leq 0.25$. However, this curve is more than 20% unconservative relative to the plastic zone solution for the pure axial loading case.

### *11.3. Key general observation*

One general observation can be made from the case study examples that is vital to an understanding of the underlying physical behaviour which produces the error $\epsilon$. For strong-axis bending, use of $K = 1$ is appropriate for design whenever distributed plasticity (i.e., residual stress) and geometric imperfection effects, not considered in the second-order elastic design analysis, have a negligible influence on the maximum strength of the frame in a sidesway mode of failure. For weak-axis bending, the shape of the AISC LRFD beam-column interaction curve accounts in part, but not completely, for the relatively severe distributed plasticity effects on the strength. However, if there is substantial distributed plasticity in the columns at any of the factored load levels, these effects can result in an additional reduction in the maximum strength, particularly in non-redundant frames. The influence of distributed plasticity and geometric imperfection effects on the sidesway strength of a steel frame is highly correlated with the elastic sidesway amplification factor $B_2$.

## 12. Conclusions

Some engineers may find it surprising that the errors in using $K = 1$ can be as large as 6% for frames in which $B_2 = 1.11$. It may be even more surprising that the errors can be as large as 16% at this $B_2$ limit if both effective length and the second-order amplification of the elastic design forces are neglected. For any given value of $B_2$, the maximum error cases involve columns with small $L/r$ and relatively small $P/P_y$ (approximately 0.17), that are restrained by heavy beams and are subjected to large leaning column effects. Columns with small $L/r$ and with $P/P_y \approx 0.17$, but which have small rotational end restraint, are nearly as critical but obtain some benefit due to column inelastic stiffness reduction. For equal values of the storey amplifier $B_2$, columns with small $L/r$ that are loaded by high axial forces also exhibit errors that are close to the maximum estimates, but these errors are slightly less than for the above two situations.

Design recommendations have been made based on the critical error cases. It is emphasized that the errors can be substantially less than the maximum estimates, particularly for cases in which there is significant redundancy, when the columns are deformed nearly in full reversed curvature bending under sidesway deflections, and for smaller rotational end restraints (for which the column inelastic stiffness reduction can be significant). Nevertheless, the use of these recommendations can substantially speed-up the design calculations, and will lead to a design close to that based on explicit calculation of effective lengths.

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