

DYNAMIC ANALYSIS BY NUMERICAL INTEGRATION

*Normally, For Earthquake Loading
Direct Numerical Integration Is Very Slow*

20.1 INTRODUCTION

The most general approach for the solution of the dynamic response of structural systems is the direct numerical integration of the dynamic equilibrium equations. This involves, after the solution is defined at time zero, the attempt to satisfy dynamic equilibrium at discrete points in time. Most methods use equal time intervals at Δt , $2\Delta t$, $3\Delta t$ $N\Delta t$. Many different numerical techniques have previously been presented; however, all approaches can fundamentally be classified as either *explicit* or *implicit* integration methods.

Explicit methods do not involve the solution of a set of linear equations at each step. Basically, these methods use the differential equation at time “ t ” to predict a solution at time “ $t + \Delta t$ ”. For most real structures, which contain stiff elements, a very small time step is required in order to obtain a stable solution. Therefore, all explicit methods are *conditionally stable* with respect to the size of the time step.

Implicit methods attempt to satisfy the differential equation at time “ t ” after the solution at time “ $t - \Delta t$ ” is found. These methods require the solution of a set of linear equations at each time step; however, larger time steps may be used. Implicit methods can be *conditionally or unconditionally stable*.

There exist a large number of accurate, higher-order, multi-step methods that have been developed for the numerical solution of differential equations. These multi-step methods assume that the solution is a smooth function in which the higher derivatives are continuous. The exact solution of many nonlinear structures requires that the accelerations, the second derivative of the displacements, are not smooth functions. This discontinuity of the acceleration is caused by the nonlinear hysteresis of most structural materials, contact between parts of the structure, and buckling of elements. Therefore, only single-step methods will be presented in this chapter. Based on a significant amount of experience, it is the conclusion of the author that only single-step, implicit, unconditional stable methods be used for the step-by-step seismic analysis of practical structures.

20.2 NEWMARK FAMILY OF METHODS

In 1959 Newmark [1] presented a family of single-step integration methods for the solution of structural dynamic problems for both blast and seismic loading. During the past 40 years Newmark's method has been applied to the dynamic analysis of many practical engineering structures. In addition, it has been modified and improved by many other researchers. In order to illustrate the use of this family of numerical integration methods consider the solution of the linear dynamic equilibrium equations written in the following form:

$$\mathbf{M}\ddot{\mathbf{u}}_t + \mathbf{C}\dot{\mathbf{u}}_t + \mathbf{K}\mathbf{u}_t = \mathbf{F}_t \quad (20.1)$$

The direct use of Taylor's series provides a rigorous approach to obtain the following two additional equations:

$$\mathbf{u}_t = \mathbf{u}_{t-\Delta t} + \Delta t \dot{\mathbf{u}}_{t-\Delta t} + \frac{\Delta t^2}{2} \ddot{\mathbf{u}}_{t-\Delta t} + \frac{\Delta t^3}{6} \dddot{\mathbf{u}}_{t-\Delta t} + \dots \quad (20.2a)$$

$$\dot{\mathbf{u}}_t = \dot{\mathbf{u}}_{t-\Delta t} + \Delta t \ddot{\mathbf{u}}_{t-\Delta t} + \frac{\Delta t^2}{2} \dddot{\mathbf{u}}_{t-\Delta t} + \dots \quad (20.2b)$$

Newmark truncated these equations and expressed them in the following form:

$$\mathbf{u}_t = \mathbf{u}_{t-\Delta t} + \Delta t \dot{\mathbf{u}}_{t-\Delta t} + \frac{\Delta t^2}{2} \ddot{\mathbf{u}}_{t-\Delta t} + \beta \Delta t^3 \ddot{\mathbf{u}} \quad (20.2a)$$

$$\dot{\mathbf{u}}_t = \dot{\mathbf{u}}_{t-\Delta t} + \Delta t \ddot{\mathbf{u}}_{t-\Delta t} + \gamma \Delta t^2 \ddot{\mathbf{u}} \quad (20.2b)$$

If the acceleration is assumed to be linear within the time step, the following equation can be written:

$$\ddot{\mathbf{u}} = \frac{(\ddot{\mathbf{u}}_t - \ddot{\mathbf{u}}_{t-\Delta t})}{\Delta t} \quad (20.3)$$

The substitution of Equation (20.3) into Equations (20.2a and b) produces Newmark's equations in standard form

$$\mathbf{u}_t = \mathbf{u}_{t-\Delta t} + \Delta t \dot{\mathbf{u}}_{t-\Delta t} + \left(\frac{1}{2} - \beta\right) \Delta t^2 \ddot{\mathbf{u}}_{t-\Delta t} + \beta \Delta t^2 \ddot{\mathbf{u}}_t \quad (20.4a)$$

$$\dot{\mathbf{u}}_t = \dot{\mathbf{u}}_{t-\Delta t} + (1 - \gamma) \Delta t \ddot{\mathbf{u}}_{t-\Delta t} + \gamma \Delta t \ddot{\mathbf{u}}_t \quad (20.4b)$$

Newmark used Equations (20.4a, 20.4b and 20.1) iteratively, for each time step, for each displacement DOF of the structural system. The term \ddot{u}_t was obtained from Equation (20.1) by dividing the equation by the mass associated with the DOF.

In 1962 Wilson [2] formulated Newmark's method in matrix notation, added stiffness and mass proportional damping, and eliminated the need for iteration by introducing the direct solution of equations at each time step. This requires that Equations (20.4a and 20.4b) be rewritten in the following form:

$$\ddot{\mathbf{u}}_t = b_1(\mathbf{u}_t - \mathbf{u}_{t-\Delta t}) + b_2 \dot{\mathbf{u}}_{t-\Delta t} + b_3 \ddot{\mathbf{u}}_{t-\Delta t} \quad (20.5a)$$

$$\dot{\mathbf{u}}_t = b_4(\mathbf{u}_t - \mathbf{u}_{t-\Delta t}) + b_5 \dot{\mathbf{u}}_{t-\Delta t} + b_6 \ddot{\mathbf{u}}_{t-\Delta t} \quad (20.5b)$$

where the constants b_1 to b_6 are defined in Table 20.1. The substitution of Equations (20.5a and 20.5b) into Equation (20.1) allows the dynamic equilibrium of the system at time "t" to be written in terms of the unknown node displacements \mathbf{u}_t . Or,

$$(b_1\mathbf{M} + b_4\mathbf{C} + \mathbf{K})\mathbf{u}_t = \mathbf{F}_t + \mathbf{M}(b_1\mathbf{u}_{t-\Delta t} - b_2\dot{\mathbf{u}}_{t-\Delta t} - b_3\ddot{\mathbf{u}}_{t-\Delta t}) + \mathbf{C}(b_4\mathbf{u}_{t-\Delta t} - b_5\dot{\mathbf{u}}_{t-\Delta t} - b_6\ddot{\mathbf{u}}_{t-\Delta t}) \quad (20.6)$$

The Newmark direct integration algorithm is summarized in Table 20.1. Note that the constants b_i need be calculated only once. Also, for linear systems, the effective dynamic stiffness matrix $\bar{\mathbf{K}}$ is formed and triangularized only once.

20.3 STABILITY OF NEWMARK'S METHOD

For zero damping Newmark's method is conditionally stable if

$$\gamma \geq \frac{1}{2}, \quad \beta \leq \frac{1}{2} \quad \text{and} \quad \Delta t \leq \frac{1}{\omega_{\text{MAX}} \sqrt{\gamma/2 - \beta}} \quad (20.7)$$

where ω_{MAX} is the maximum frequency in the structural system [1]. Newmark's method is unconditionally stable if

$$2\beta \geq \gamma \geq \frac{1}{2} \quad (20.8)$$

However, if γ is greater than $1/2$, errors are introduced. These errors are associated with "numerical damping" and "period elongation".

For large multi degree-of-freedom structural systems the time step limit, given by Equation (20.7), can be written in a more useable form as

$$\frac{\Delta t}{T_{\text{MIN}}} \leq \frac{1}{2\pi \sqrt{\gamma/2 - \beta}} \quad (20.9)$$

Computer models of large real structures normally contain a large number of periods which are smaller than the integration time step; therefore, it is essential that one select a numerical integration method that is unconditional for all time steps.

Table 20.1. Summary of the Newmark Method for Direct Integration**I. INITIAL CALCULATION**

A. Form static stiffness matrix \mathbf{K} , mass matrix \mathbf{M} and damping matrix \mathbf{C}

B. Specify integration parameters β and γ

C. Calculate integration constants

$$b_1 = \frac{1}{\beta \Delta t^2} \quad b_2 = \frac{1}{\beta \Delta t} \quad b_3 = \beta - \frac{1}{2} \quad b_4 = \gamma \Delta t b_1$$

$$b_5 = 1 + \gamma \Delta t b_2 \quad b_6 = \Delta t (1 + \gamma b_3 - \gamma)$$

D. Form effective stiffness matrix $\bar{\mathbf{K}} = \mathbf{K} + b_1 \mathbf{M} + b_4 \mathbf{C}$

E. Triangularize effective stiffness matrix $\bar{\mathbf{K}} = \mathbf{LDL}^T$

F. Specify initial conditions \mathbf{u}_0 , $\dot{\mathbf{u}}_0$, $\ddot{\mathbf{u}}_0$

II. FOR EACH TIME STEP $t = \Delta t, 2\Delta t, 3\Delta t$ -----

A. Calculate effective load vector

$$\bar{\mathbf{F}}_t = \mathbf{F}_t + \mathbf{M}(b_1 \mathbf{u}_{t-\Delta t} - b_2 \dot{\mathbf{u}}_{t-\Delta t} - b_3 \ddot{\mathbf{u}}_{t-\Delta t}) + \mathbf{C}(b_4 \mathbf{u}_{t-\Delta t} - b_5 \dot{\mathbf{u}}_{t-\Delta t} - b_6 \ddot{\mathbf{u}}_{t-\Delta t})$$

B. Solve for node displacement vector at time t

$$\mathbf{LDL}^T \mathbf{u}_t = \bar{\mathbf{F}}_t \quad \text{forward and back-substitution only}$$

C. Calculate node velocities and accelerations at time t

$$\dot{\mathbf{u}}_t = b_4 (\mathbf{u}_t - \mathbf{u}_{t-\Delta t}) + b_5 \dot{\mathbf{u}}_{t-\Delta t} + b_6 \ddot{\mathbf{u}}_{t-\Delta t}$$

$$\ddot{\mathbf{u}}_t = b_1 (\mathbf{u}_t - \mathbf{u}_{t-\Delta t}) + b_2 \dot{\mathbf{u}}_{t-\Delta t} + b_3 \ddot{\mathbf{u}}_{t-\Delta t}$$

D. Go to Step II.A with $t = t + \Delta t$

20.4 THE AVERAGE ACCELERATION METHOD

The average acceleration method is identical to the trapezoidal rule that has been used to numerically evaluate second order differential equations for approximately 100 years. It can easily be derived from the following truncated Taylor's series expansion:

$$\begin{aligned} \mathbf{u}_\tau &= \mathbf{u}_{t-\Delta t} + \tau \dot{\mathbf{u}}_{t-\Delta t} + \frac{\tau^2}{2} \ddot{\mathbf{u}}_{t-\Delta t} + \frac{\tau^3}{6} \dddot{\mathbf{u}}_{t-\Delta t} + \dots \\ &\approx \mathbf{u}_{t-\Delta t} + \tau \dot{\mathbf{u}}_{t-\Delta t} + \frac{\tau^2}{2} \left(\frac{\ddot{\mathbf{u}}_{t-\Delta t} + \ddot{\mathbf{u}}_t}{2} \right) \end{aligned} \quad (20.10)$$

where τ is a variable point within the time step. The consistent velocity can be obtained by differentiation of Equation (20.10). Or,

$$\dot{\mathbf{u}}_\tau = \dot{\mathbf{u}}_{t-\Delta t} + \tau \left(\frac{\ddot{\mathbf{u}}_{t-\Delta t} + \ddot{\mathbf{u}}_t}{2} \right) \quad (20.11)$$

If $\tau = \Delta t$

$$\mathbf{u}_t = \mathbf{u}_{t-\Delta t} + \Delta t \dot{\mathbf{u}}_{t-\Delta t} + \frac{\Delta t^2}{4} \ddot{\mathbf{u}}_{t-\Delta t} + \frac{\Delta t^2}{4} \ddot{\mathbf{u}}_t \quad (20.12a)$$

$$\dot{\mathbf{u}}_t = \dot{\mathbf{u}}_{t-\Delta t} + \frac{\Delta t}{2} \ddot{\mathbf{u}}_{t-\Delta t} + \frac{\Delta t}{2} \ddot{\mathbf{u}}_t \quad (20.12b)$$

These equations are identical to Newmark's Equations (20.4a and b) with $\gamma = 1/2$ and $\beta = 1/4$.

It can easily be shown that the average acceleration method conserves energy for the free vibration problem, $\mathbf{M}\ddot{\mathbf{u}} + \mathbf{K}\mathbf{u} = \mathbf{0}$, for all possible time steps [4]. . Therefore, the sum of the kinetic and strain energy is constant. Or,

$$2E = \dot{\mathbf{u}}_t^T \mathbf{M} \dot{\mathbf{u}}_t + \mathbf{u}_t^T \mathbf{K} \mathbf{u}_t = \dot{\mathbf{u}}_{t-\Delta t}^T \mathbf{M} \dot{\mathbf{u}}_{t-\Delta t} + \mathbf{u}_{t-\Delta t}^T \mathbf{K} \mathbf{u}_{t-\Delta t} \quad (20.13)$$

20.5 WILSON'S θ FACTOR

In 1973, the general Newmark method was made unconditionally stable by the introduction of a θ factor [3]. The introduction of the θ factor is motivated by the observation that an unstable solution tends to oscillate about the true solution. Therefore, if the numerical solution is evaluated within the time increment the spurious oscillations are minimized. This can be accomplished by a simple modification to the Newmark method by using a time step defined by

$$\Delta t' = \theta \Delta t \quad (20.14a)$$

and a load defined by

$$\mathbf{R}_{t'} = \mathbf{R}_{t-\Delta t} + \theta (\mathbf{R}_t - \mathbf{R}_{t-\Delta t}) \quad (20.14b)$$

where $\theta \geq 1.0$. After the acceleration $\ddot{\mathbf{u}}_{t'}$ vector is evaluated by Newmark's method at the integration time step $\theta \Delta t$, values of node accelerations, velocities and displacements are calculated from the following fundamental equations:

$$\ddot{\mathbf{u}}_t = \ddot{\mathbf{u}}_{t-\Delta t} + \frac{1}{\theta} (\ddot{\mathbf{u}}_{t'} - \ddot{\mathbf{u}}_{t-\Delta t}) \quad (20.15a)$$

$$\dot{\mathbf{u}}_t = \dot{\mathbf{u}}_{t-\Delta t} + (1 - \gamma) \Delta t \ddot{\mathbf{u}}_{t-\Delta t} + \gamma \Delta t \ddot{\mathbf{u}}_t \quad (20.15b)$$

$$\mathbf{u}_t = \mathbf{u}_{t-\Delta t} + \Delta t \dot{\mathbf{u}}_{t-\Delta t} + \frac{\Delta t^2 (1 - 2\beta)}{2} \ddot{\mathbf{u}}_{t-\Delta t} + \beta \Delta t^2 \ddot{\mathbf{u}}_t \quad (20.15c)$$

The use of the θ factor tends to numerically damp out the high modes of the system. If θ equals 1.0 Newmark's method is not modified. However, for problems where the higher mode response is important, the errors that are introduced can be large. In addition, the dynamic equilibrium equations are not exactly satisfied at time t . Therefore, the author no longer recommends the use of the θ factor. At the time of the introduction of the method, it solved all problems associated with stability of the Newmark family of methods. However, during the past twenty years new and more accurate numerical methods have been developed.

20.6 THE USE OF STIFFNESS PROPORTIONAL DAMPING

Because of the unconditional stability of the average acceleration method, it is the most robust method to be used for the step-by-step dynamic analysis of large complex structural systems in which a large number of high frequencies, short periods, are present. The only problem with the method is that the short periods, which are smaller than the time step, oscillate indefinitely after they are excited. The higher mode oscillation can be reduced by the addition of stiffness proportional damping. The additional damping that is added to the system is of the form

$$C_D = \delta K \quad (20.16)$$

where the modal damping ratio, given by Equation (8.8) is defined by

$$\xi_n = \frac{1}{2} \delta \omega_n = \frac{\pi}{T_n} \delta \quad (20.17)$$

One notes that the damping is large for short periods and small for the long periods or low frequencies. It is apparent that periods which are greater than the time step cannot be integrated accurately by any direct integration method. Therefore, it is logical to damp these short periods to prevent them from oscillating during the solution procedure. For a time step equal to the period, Equation (20.17) can be rewritten as

$$\delta = \xi_n \frac{\Delta T}{\pi} \quad (20.18)$$

Hence, if the integration time step is 0.02 seconds and we wish to assign a minimum of 1.0 to all periods shorter than the time step, a value of $\delta = 0.0064$ should be used. The damping ratio in all modes is now predictable for this example from Equation (20.17). Therefore, the damping ratio for a 1.0 second period is 0.02 and for a 0.10 second period is 0.2.

20.7 THE HILBER, HUGHES AND TAYLOR α METHOD

The α method [4] uses the Newmark method to solve the following modified equations of motion:

$$\mathbf{M}\ddot{\mathbf{u}}_t + (1 + \alpha) \mathbf{C}\dot{\mathbf{u}}_t + (1 + \alpha) \mathbf{K}\mathbf{u}_t = (1 + \alpha)\mathbf{F}_t - \alpha\mathbf{F}_t + \alpha\mathbf{C}\dot{\mathbf{u}}_{t-\Delta t} + \alpha\mathbf{K}\mathbf{u}_{t-\Delta t} \quad (20.19)$$

With α equals zero the method reduces to the constant acceleration method. It produces numerical energy dissipation in the higher modes; however, it cannot be predicted as a damping ratio as in the use of stiffness proportional damping. Also, it does not solve the fundamental equilibrium equation at time t . However, it is currently being used in many computer programs. The performance of the method appears to be very similar to the use of stiffness proportional damping.

20.8 SELECTION OF A DIRECT INTEGRATION METHOD

It is apparent that a large number of different direct numerical integration methods are possible by specifying different integration parameters. A few of the most commonly used methods are summarized in 20.2.

Table 20.2. Summary of Newmark Methods Modified by the δ Factor

METHOD	γ	β	δ	$\frac{\Delta t}{T_{\text{MIN}}}$	ACCURACY
Central Difference	1/2	0	0	0.3183	Excellent for small Δt Unstable for large Δt
Linear Acceleration	1/2	1/6	0	0.5513	Very good for small Δt Unstable for large Δt
Average Acceleration	1/2	1/4	0	∞	Good for small Δt No energy dissipation
Modified Average Acceleration	1/2	1/4	$\frac{\Delta T}{\pi}$	∞	Good for small Δt Energy dissipation for large Δt

For single degree-of-freedom systems the central difference method is most accurate; and the linear acceleration method is more accurate than the average acceleration method. However, if only single degree-of-freedom systems are to be integrated the piece-wise exact method, previously presented, should be used since there is no need to use an approximate method.

It appears that the modified average acceleration method, with a minimum addition of stiffness proportional damping, is a general procedure that can be used for the dynamic analysis of all structural systems. Using $\delta = \Delta T / \pi$ will damp out periods shorter than the time step and introduces a minimum error in the long period response.

20.9 NONLINEAR ANALYSIS

The basic Newmark Constant acceleration method can be extended to nonlinear dynamic analysis. This requires that iteration must be performed at each time step in order to satisfy equilibrium. Also, the incremental stiffness matrix must be formed and triangularized at each iteration or at selective points in time. Many different numerical tricks, including element by element methods, have been developed in order to minimize the computational requirements. Also, the triangularization of the effective incremental stiffness matrix may be avoided by the introduction of iterative solution methods.

20.10 SUMMARY

For earthquake analysis of linear structures, it should be noted that the direct integration of the dynamic equilibrium equations is normally not numerically efficient as compared to the mode superposition method using LDR vectors. If the triangularized stiffness and mass matrices and other vectors cannot be stored in high-speed storage, the computer execution time can be large.

After using direct integration methods for approximately forty years, the author can no longer recommend the Wilson method for the direct integration of the dynamic equilibrium equations. The Newmark constant acceleration method, with the addition of very small amount of stiffness proportional damping, is recommended for dynamic analysis nonlinear structural systems. For all methods of direct

integration great care should be taken to make certain that the stiffness proportional damping does not eliminate important high-frequency response. Mass proportional damping cannot be justified because it causes external forces to be applied to the structure that reduce the base shear for seismic loading.

In the area of nonlinear dynamic analysis one cannot prove that any one method will always converge. One should always check the error in the conservation of energy for every solution obtained. In future editions of this book it is hoped that numerical examples will be presented in order that the appropriate method can be recommended for different classes of problems in structural analysis.

20.11 REFERENCES

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