DYNAMIC ANALYSIS USING MODE SUPERPOSITION

The Mode Shapes Used To Uncouple The Dynamic Equilibrium Equations Need Not Be The Exact Free-Vibration Mode Shapes

13.1 EQUATIONS TO BE SOLVED

The dynamic force equilibrium Equation (12.4) can be rewritten in the following form as a set of N second order differential equations:

$$\mathbf{M}\ddot{\mathbf{u}}(t) + \mathbf{C}\dot{\mathbf{u}}(t) + \mathbf{K}\mathbf{u}(t) = \mathbf{F}(t) = \sum_{j=1}^{J} \mathbf{f}_j \, \mathbf{g}(t)_j \qquad (13.1)$$

All possible types of time-dependent loading, including wind, wave and seismic, can be represented by a sum of "J" space vectors $\mathbf{f}_j$, which are not a function of time, and J time functions $\mathbf{g}(t)_j$, where J cannot be greater than the number of displacements N.

The number of dynamic degrees-of-freedom is equal to the number of lumped masses in the system. Many publications advocate the elimination of all massless displacements by static condensation prior to the solution of Equation (13.1). The static condensation method reduces the number of dynamic equilibrium equations to solve; however, it can significantly increase the density and the bandwidth of the condensed stiffness matrix. In building type structures, in which each diaphragm has only three lumped masses, this approach is effective and is automatically used in building analysis programs.

For the dynamic solution of arbitrary structural systems, however, the elimination of the massless displacement is, in general, not numerically efficient. Therefore, the modern versions of the SAP program do not use static condensation in order to retain the sparseness of the stiffness matrix.

13.2 TRANSFORMATION TO MODAL EQUATIONS

The fundamental mathematical method that is used to solve Equation (13.1) is the separation of variables. This approach assumes the solution can be expressed in the following form:

$$\mathbf{u}(t) = \Phi\mathbf{Y}(t) \tag{13.2a}$$

Where Φ is an "N by L" matrix containing L spatial vectors which are not a function of time, and $\mathbf{Y}(t)$ is a vector containing L functions of time.

From Equation (13.2a) it follows that

$$\dot{\mathbf{u}}(t) = \Phi\dot{\mathbf{Y}}(t) \quad \text{and} \quad \ddot{\mathbf{u}}(t) = \Phi\ddot{\mathbf{Y}}(t) \qquad \text{(13.2b) and (13.2c)}$$

Prior to solution, we require that the space functions satisfy the following mass and stiffness orthogonality conditions:

$$\Phi^{T}M\Phi = I \quad \text{and} \quad \Phi^{T}K\Phi = \Omega^{2} \tag{13.3}$$

where $\mathbf{I}$ is a diagonal unit matrix and Ω^2 is a diagonal matrix which may or may not contain the free vibration frequencies. It should be noted that the fundamentals of mathematics place no restrictions on these vectors, other than the orthogonality properties. In this book all space function vectors are normalized so that the *Generalized Mass* $\phi_n^{\,T} M\phi_n = 1$.

After substitution of Equations (13.2) into Equation (13.1) and the pre-multiplication by Φ^{T}, the following matrix of L equations are produced:

$$\mathbf{I}\ddot{\mathbf{Y}}(t) + \mathbf{d}\dot{\mathbf{Y}}(t) + \Omega^2 = \sum_{j=1}^{J} \mathbf{p}_j \mathbf{g}(t)_j \tag{13.4}$$

where $\mathbf{p}_j = \Phi^T \mathbf{f}_j$ and are defined as the modal participation factors for time function j. The term p_{nj} is associated with the n th mode.

For all real structures the "L x L" matrix **d** is not diagonal; however, in order to uncouple the modal equations it is necessary to assume that there is no coupling between the modes. Therefore, it is assumed to be diagonal with the modal damping terms defined by

$$d_{nn} = 2\zeta_n \omega_n \tag{13.5}$$

where ζ_n is defined as the ratio of the damping in mode n to the critical damping of the mode [1].

A typical uncoupled modal equation, for linear structural systems, is of the following form:

$$\ddot{y}(t)_n + 2\zeta_n \omega_n \dot{y}(t)_n + \omega_n^2 y(t)_n = \sum_{j=1}^{J} p_{nj}\, g(t)_j \tag{13.6}$$

For three dimensional seismic motion, this equation can be written as

$$\ddot{y}(t)_n + 2\zeta_n \omega_n \dot{y}(t)_n + \omega_n^2 y(t)_n = p_{nx}\ddot{u}(t)_{gx} + p_{ny}\ddot{u}(t)_{gy} + p_{nz}\ddot{u}(t)_{gz} \tag{13.7}$$

where the three directional *Mass Participation Factors* are defined by $p_{ni} = -\phi_n^T \mathbf{M}_i$ in which *i* is equal to x, y or z and *n* is the mode number.

Prior to presenting the solution of Equation (13.6) for various types of loading it is convenient to define additional constants and functions which are summarized in Table 13.1. This will allow many of the equations presented in other parts of this book to be written in a compact form. Also, the notation reduces the tedium involved in the algebraic derivation and verification of various equations. In addition, it will allow the equations to be in a form that can be easily programmed and verified.

13.3 RESPONSE DUE TO INITIAL CONDITIONS ONLY

If the "n" subscript is dropped, Equation (13.6) can be written for a typical mode as

$$\ddot{y}(t) + 2\xi\omega\dot{y}(t) + \omega^2 y(t) = 0 \tag{13.8}$$

in which the initial modal displacement y_0 and velocity $\dot{y}_0$ are specified due to previous loading acting on the structure. Note that the functions $S(t)$ and $C(t)$, given in Table 13.1, are solutions to Equation (13.8).

Table 13.1. Summary of Notation used in Dynamic Response Equations

CONSTANTS

$$\omega_D = \omega\sqrt{1-\xi^2} \qquad \overline{\omega} = \omega\xi \qquad \overline{\xi} = \frac{\xi}{\sqrt{1-\xi^2}} \qquad a_0 = \frac{2\xi}{\omega\Delta t}$$

$$a_1 = 1 + a_0 \qquad a_2 = -\frac{1}{\Delta t} \qquad a_3 = -\overline{\xi}a_1 - a_2/\omega_D \qquad a_4 = -a_1$$

$$a_5 = -a_0 \qquad a_6 = -a_2 \qquad a_7 = -\overline{\xi}a_5 - a_6/\omega_D \qquad a_8 = -a_5$$

$$a_9 = \omega_D^2 - \overline{\omega}^2 \qquad a_{10} = 2\overline{\omega}\omega_D$$

FUNCTIONS

$$S(t) = e^{-\xi\omega t}\sin(\omega_D t) \qquad \dot{S}(t) = -\overline{\omega}S(t) + \omega_D C(t)$$

$$C(t) = e^{-\xi\omega t}\cos(\omega_D t) \qquad \dot{C}(t) = -\overline{\omega}C(t) - \omega_D S(t)$$

$$A_1(t) = C(t) + \overline{\xi}S(t) \qquad \ddot{S}(t) = -a_9 S(t) - a_{10}C(t)$$

$$A_2(t) = \frac{1}{\omega_D}S(t) \qquad \ddot{C}(t) = -a_9 C(t) + a_{10}S(t)$$

$$A_3(t) = \frac{1}{\omega^2}[a_1 + a_2 t + a_3 S(t) + a_4 C(t)]$$

$$A_4(t) = \frac{1}{\omega^2}[a_5 + a_6 t + a_7 S(t) + a_8 C(t)]$$

The solution of Equation (13.8) can now be written in the following compact form:

$$y(t) = A_1(t)y_0 + A_2(t)\dot{y}_0 \tag{13.9}$$

This solution can be easily verified since it satisfies Equation (13.8) and the initial conditions.

13.4 GENERAL SOLUTION DUE TO ARBITRARY LOADING

There are many different methods available to solve the typical modal equations. However, the use of the exact solution for a linear load over a small time increment has been found to be the most economical and accurate method to numerically solve this equation within computer programs. It does not have problems with stability and it does not introduce numerical damping. Since most seismic ground motions are defined as linear within 0.005 second intervals, the method is exact for this type of loading for all frequencies.

In order to simplify the notation, all loads are added together to form a typical modal equation of the following form:

$$\ddot{y}(t) + 2\zeta\omega\dot{y}(t) + \omega^2 y(t) = R(t) \tag{13.10}$$

where the modal loading $R(t)$ is a piece-wise linear function as shown in Figure 13.1.

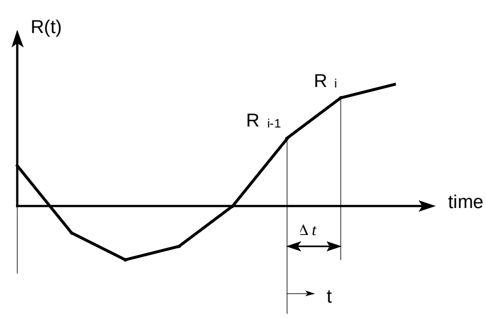

Figure 13.1 Typical Modal Load Function

The equation for the linear load function within the time step is by definition

$$R(t) = (1 - \frac{t}{\Delta t})R_{i-1} + \frac{t}{\Delta t}R_i \tag{13.11}$$

where the time t is in reference to the start of the time step. Now the exact solution within the time step can be written as

$$y(t) = A_1(t)y_{i-1} + A_2(t)\dot{y}_{i-1} + A_3(t)R_{i-1} + A_4(t)R_i \tag{13.12a}$$

where all functions are defined in Table 13.1. Again, the solution can be easily verified by substitution of Equation (13.12a) into Equation (13.10). It is apparent that the exact modal velocity and acceleration within the time step are given by

$$\dot{y}(t) = \dot{A}_1(t)y_{i-1} + \dot{A}_2(t)\dot{y}_{i-1} + \dot{A}_3(t)R_{i-1} + \dot{A}_4(t)R_i \tag{13.12b}$$

$$\ddot{y}(t) = \ddot{A}_1(t)y_{i-1} + \ddot{A}_2(t)\dot{y}_{i-1} + \ddot{A}_3(t)R_{i-1} + \ddot{A}_4(t)R_i \tag{13.12c}$$

Equations (13.12a, b and c) are evaluated at the end of the time increment Δt and the following modal displacement, velocity and acceleration at the end of the i th time step are given by the following set of recurrence equations:

$$y_i = A_1\, y_{i-1} + A_2\, \dot{y}_{i-1} + A_3\, R_{i-1} + A_4\, R_i \tag{13.13a}$$

$$\dot{y}_i = A_5\, y_{i-1} + A_6\, \dot{y}_{i-1} + A_7\, R_{i-1} + A_8\, R_i \tag{13.13b}$$

$$\ddot{y}_i = A_9\, y_{i-1} + A_{10}\, \dot{y}_{i-1} + A_{11}\, R_{i-1} + A_{12}\, R_i \tag{13.13c}$$

The constants A_1 to A_{12}, which are summarized in Table 13.2, need to be computed only once for each mode. Therefore, for each time increment only 12 multiplications and 9 additions are required. Modern, inexpensive personal computers can complete one multiplication and one addition in approximately 10^{-6} seconds. Hence, the computer time required to solve for 200 steps per second for a 50 second duration earthquake is approximately 0.01 seconds. Or, 100 modal equations can be solved in one second of computer time. Therefore, there is no need to consider other numerical methods, such as the approximate Fast Fourier Transformation method or the numerical evaluation of the Duhamel integral, to solve these equations. Because of the speed of this exact piece-wise linear technique, it can also be used to develop accurate earthquake response spectra using a very small amount of computer time.

Table 13.2. Constants Used in Recurrence Equations (13.13)

$$A_1 = A_1(\Delta t) = C(\Delta t) + \overline{\xi}\, S(\Delta t)$$

$$A_2 = A_2(\Delta t) = \frac{1}{\omega_D} S(\Delta t)$$

$$A_3 = A_3(\Delta t) = \frac{1}{\omega^2}[a_1 + a_2\Delta t + a_3 S(\Delta t) + a_4 C(\Delta t)]$$

$$A_4 = A_4(\Delta t) = \frac{1}{\omega^2}[a_5 + a_6\Delta t + a_7 S(\Delta t) + a_8 C(\Delta t)]$$

$$A_5 = \dot{A}_1(\Delta t) = \dot{C}(\Delta t) + \overline{\xi}\, \dot{S}(\Delta t)$$

$$A_6 = \dot{A}_2(\Delta t) = \frac{1}{\omega_D} \dot{S}(\Delta t)$$

$$A_7 = \dot{A}_3(\Delta t) = \frac{1}{\omega^2}[a_2 + a_3\dot{S}(\Delta t) + a_4\dot{C}(\Delta t)]$$

$$A_8 = \dot{A}_4(\Delta t) = \frac{1}{\omega^2}[a_6 + a_7\dot{S}(\Delta t) + a_8\dot{C}(\Delta t)]$$

$$A_9 = \ddot{A}_1(\Delta t) = \ddot{C}(\Delta t) + \overline{\xi}\, \ddot{S}(\Delta t)$$

$$A_{10} = \ddot{A}_2(\Delta t) = \frac{1}{\omega_D} \ddot{S}(\Delta t)$$

$$A_{11} = \ddot{A}_3(\Delta t) = \frac{1}{\omega^2}[a_3\ddot{S}(\Delta t) + a_4\ddot{C}(\Delta t)]$$

$$A_{12} = \ddot{A}_4(\Delta t) = \frac{1}{\omega^2}[a_7\ddot{S}(\Delta t) + a_8\ddot{C}(\Delta t)]$$

13.5 SOLUTION FOR PERIODIC LOADING

The recurrence solution algorithm, summarized by Equation 13.13, is a very efficient computational method for arbitrary, transient, dynamic loads with initial conditions. It is possible to use this same simple solution method for arbitrary periodic loading as shown in Figure 13.2. Note that the total duration of the loading is from $-\infty$ to $+\infty$ and the loading function has the same amplitude and shape for each typical period T_p. Wind, sea wave and acoustic forces can produce this type of periodic loading. Also, dynamic live loads on bridges may also be of periodic form.

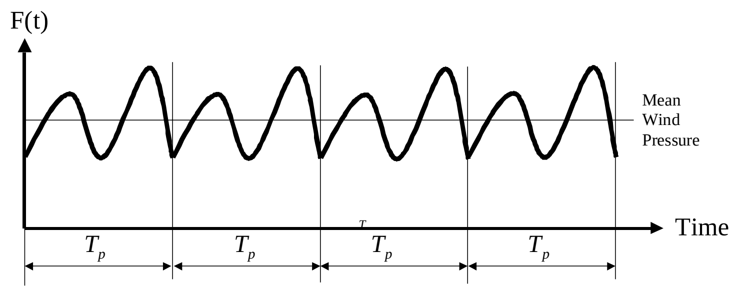

Figure 13.2. Example of Periodic Loading

For a typical duration T_p of loading, a numerical solution, for each mode, can be evaluated by the application of Equation (13.13) without initial conditions. This solution is incorrect since it does not have the correct initial conditions. Therefore, it is necessary for this solution $y(t)$ to be corrected in order that the exact solution $z(t)$ has the same displacement and velocity at the beginning and end of each loading period. In order to satisfy the basic dynamic equilibrium equation the corrective solution $x(t)$ must have the following form:

$$x(t) = x_0 A_1(t) + \dot{x}_0 A_2(t) \tag{13.14}$$

where the functions are defined in Table 13.1.

The total exact solution for displacement and velocity for each mode can now be written as

$$z(t) = y(t) + x(t) \tag{13.15a}$$

$$\dot{z}(t) = \dot{y}(t) + \dot{x}(t) \tag{13.15b}$$

In order that the exact solution is periodic the following conditions must be satisfied:

$$z(T_p) = z(0) \tag{13.16a}$$

$$\dot{z}(T_p) = \dot{z}(0) \tag{13.16b}$$

The numerical evaluation of Equation (13.14) produces the following matrix equation which must be solved for the unknown initial conditions:

$$\begin{bmatrix} 1 - A_1(T_p) & -A_2(T_p) \\ -\dot{A}_1(T_p) & 1 - \dot{A}_2(T_p) \end{bmatrix} \begin{bmatrix} x_0 \\ \dot{x}_0 \end{bmatrix} = \begin{bmatrix} -y(T_p) \\ -\dot{y}(T_p) \end{bmatrix} \tag{13.17}$$

The exact periodic solution for modal displacements and velocities can now be calculated from Equations (13.15a and 13.15b).

13.6 PARTICIPATING MASS RATIOS

Several Building Codes require that at least 90 percent of the participating mass is included in the calculation of response for each principal horizontal direction. This requirement is based on a unit base acceleration in a particular direction and calculating the base shear due to that load. The steady state solution for this case involves no damping or elastic forces; therefore, the modal response equations, for a unit base acceleration in the x-direction, can be written as

$$\ddot{y}_n \;=\; p_{nx} \tag{13.18}$$

The node point inertia forces, in the x-direction, for that mode are by definition

$$\mathrm{f}_{\mathrm{xn}} \;=\; \mathrm{M}\ddot{\mathrm{u}}(t) \;=\; \mathrm{M}\,\phi_n\,\ddot{\mathrm{y}}_n \;=\; p_{nx}\mathrm{M}\,\phi_n \tag{13.19}$$

The total resisting base shear in the x-direction for mode n is the sum of all node point x forces. Or,

$$V_{nx} \;=\; -\,p_{nx}\;\mathrm{I}_x^{\mathrm{T}}\mathrm{M}\,\phi_n \;=\; p_{nx}^2 \tag{13.20}$$

The total base shear in the x-direction, including L modes, will be

$$V_x = \sum_{n=1}^{L} p_{nx}^2 \tag{13.21}$$

We can now define the participating mass in all three directions as a ratio of the total mass in that direction by

$$X_{mass} = \frac{\sum_{n=1}^{L} p_{nx}^2}{\sum m_x} \tag{13.22a}$$

$$Y_{mass} = \frac{\sum_{n=1}^{L} p_{ny}^2}{\sum m_y} \tag{13.22b}$$

$$Z_{mass} = \frac{\sum_{n=1}^{L} p_{nz}^2}{\sum m_z} \tag{13.22c}$$

If all modes are used, these ratios will all be equal to 1.0. It is clear that the 90 percent participation rule is intended to estimate the accuracy of a solution for base motion only. ***It can not be used as an error estimator for other types of loading such as point loads acting on the structure***. The SAP and ETABS programs produce the contribution of each mode to these ratios. In addition, an examination of these factors gives the engineer an indication of the direction of the base shear associated with each mode.

13.7 STATIC LOAD PARTICIPATION RATIOS

For arbitrary loading it is useful to determine if the number of vectors used is adequate to approximate the true response of the structural system. One method, which the author has proposed, is to evaluate the static displacements using a truncated set of vectors to solve for the response due to static load patterns. As indicated by Equation (13.1) the loads can be written as

$$\mathbf{F}(t) = \sum_{j=1}^{J} \mathrm{f}_{\mathrm{j}}\, \mathrm{g}(\mathrm{t})_{\mathrm{j}} \tag{13.23}$$

If one solves the statics problem for the exact displacement u_j due to the load pattern f_j the total strain energy associated with load condition j is

$$E_j = \frac{1}{2} f_j^T u_j \tag{13.24}$$

From the fundamental definition of the mode superposition method, a truncated set of vectors defines the approximate static displacement $\overline{u}_j$ as

$$\overline{u}_j = \sum_{n=1}^{L} y_n \phi_n \tag{13.25}$$

where, from Equation 13.6, the static modal response, neglecting inertia and damping forces, is given by

$$y_n = \frac{1}{\omega_n^{\ 2}} \phi_n^T f_j \tag{13 .26}$$

The total strain energy associated with the truncated mode shape solution is

$$\overline{E}_j = \frac{1}{2} f_j^T \overline{u}_j = \frac{1}{2} \sum_{n=1}^{L} \left(\frac{\phi_n^T f_j}{\omega_n} \right)^2 \tag{13.27}$$

A ***load participation ratio*** r_j can now be defined for load condition j as

$$r_j = \frac{\overline{E}_j}{E_j} \tag{13.28}$$

If this ratio is close to 1.0 the errors introduced by vector truncation will be very small. However, if this ratio is less than 90 percent additional vectors should be used in the analysis in order to capture the ***static load response***. Additional experience with this factor is required in order to use it as an accurate error estimator for all problems.

It has been the experience of the author that the use of exact eigenvectors is not an accurate vector basis for the dynamic analysis of structures subjected to point loads. Whereas, load-dependent vectors, which are defined in the following chapter, always produce a load participation ratio of 1.0.

13.8 SUMMARY

The mode superposition method is a very powerful method used to reduce the number of unknowns in a dynamic response analysis. All types of loading can be accurately approximated by piece-wise linear functions within a small time increment. An exact solution exists for this type of loading and this solution can be computed with a trivial amount of computer time for equal time increments. Therefore, there is no need to present other methods for the numerical evaluation of modal equations when a computer program is used.

To solve for the linear dynamic response of structures subjected to periodic loading it is only necessary to add a corrective solution to the transient solution for a typical time period of loading. Hence, only one numerical algorithm is required to solve a large number of different dynamic response problems in structural engineering.

Participating mass factors can be used to estimate the number of vectors required in an elastic seismic analysis. The use of mass participation factors to estimate the accuracy of a nonlinear seismic analysis can introduce significant errors; because, internal nonlinear forces, that are in equal and opposite directions, do not produce a base shear. A dynamic load participation ratio is defined which can be used to estimate the number of vectors required for other types of loading.